# A PRACTICAL TREATISE ON FOURIER'S THEOREM AND HARMONIC ANALYSIS

# A PRACTICAL TREATISE ON FOURIER'S THEOREM AND HARMONIC ANALYSIS FOR PHYSICISTS AND ENGINEERS

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BY

ALBERT EAGLE, B.Sc., A.R.C.Sc.

LECTURER IN MATHEMATICS IN THE VICTORIA UNIVERSITY OF MANCHESTER

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# PREFACE

In this book I have tried to give a fairly complete account of the practical methods of analysing any given curve or set of observations into its harmonic constituents, if it be a periodic function, and of detecting any periodic component in it if it is not a periodic function.

The book will, I hope, be readily intelligible, and readable, to anyone familiar with the elements of trigonometry and the calculus: for this purpose, an introductory chapter has been added dealing primarily with imaginary quantities, but including also, for convenience, a few theorems on the integrals and sums of trigonometric expressions that are required later in the book.

I have ventured to use the term "artificial function" to signify a function which is defined by one mathematical expression over one part of its range and by another expression over another part of its range. This may displease some modern writers on pure mathematics, who will contend that there is no necessity for such a term. But such functions are of such wide occurrence in nearly all branches of mathematical physics that it seems desirable to introduce such a term to cover them; and the difference between such functions and analytic ones is very obvious to all elementary students and practical workers. Fourier's theorem, in fact, may be regarded as the method *par excellence* of representing an artificial function in the *form* (or semblance) of an

analytic function. And it is partly because of this, as well as its property of analysing functions into their harmonic constituents, that it is of such great value in mathematics.

The view taken of Fourier's theorem in this work is rather different from that sometimes adopted. There are many known ways of expanding an arbitrary function in a definite series of analytic functions which expansion is valid for all points between two definite limits, and Fourier's theorem may be looked upon as merely one of such cases. In the other cases, however, we neither know nor care what the series represents outside these limits; while the Fourier series, outside these limits, represents continual repetitions of its behaviour between the limits. We have therefore taken Fourier's theorem as applying only to periodic functions, either artificial or analytic. However any function behaves over any finite range, we can, by repeating this behaviour an indefinite number of times to both minus infinity and plus infinity, obtain an artificial periodic function from it.

One advantage in looking upon Fourier's theorem in this way is that the Fourier's series (apart from the constant term) of any artificial periodic function can be directly written down from an examination of the discontinuities of the function and its successive differential coefficients. Moreover, the Fourier's series representing an *analytic* periodic function can be deduced in the same manner as the limit of the case when the discontinuities are infinitesimal; although this is rarely the easiest way of obtaining the expansion. For instance, the harmonic analysis of the function $2\cos^2 t$ can be deduced from the analysis of the artificial periodic function of period $2\pi$, and represented from 0 to $2\pi$ by $2\cos^2 qt$, by putting $q = 1$. This artificial function possesses, both for

$2\cos^2 qt$

itself and for all its differential coefficients, discontinuities when $t = 0$; and from the formulæ on p. 59 the Fourier constants are readily found to be given by

$$\pi a_n = \frac{2q \sin 4\pi q}{4q^2 - n^2}$$

and
$$\pi b_n = \frac{2n \sin^2 2\pi q}{n^2 - 4q^2}.$$

Putting $q = 1$ gives zero for all values of $n$ save when $n = 2$ for which the expressions become indeterminate. If in this case we put $q = 1 + \delta$ where $\delta$ is very small, we easily see that $a_2 = 1$ and $b_2 = 0$, and thus that the Fourier's series representing $2 \cos^2 t$ is $\cos 2t$ plus a constant.

Chapter III deals with the harmonic analysis of various simple artificial functions, for the most part built up of straight lines, parabolas or sine curves. This chapter will afford the student excellent practice in the evaluation of trigonometric integrals. I have been surprised at the very small proportion of students who can evaluate such integrals correctly after having been through an ordinary course of the Integral Calculus. This chapter accordingly will form a useful item in the student's education. In fact, the whole book has been written with the intention of making it constitute a valuable chapter in the education of any student of applied mathematics, as well as of making it valuable to the actual worker engaged in any kind of harmonic analysis.

Chapter V deals with the harmonic analysis of any periodic function defined by the co-ordinates of a number of points and will be, for many readers, the chapter of most practical importance in the book. I have tried to include therein all methods that I think of practical value, while giving full attention to

the most satisfactory ones. Attention may be drawn to what is, I think, a new method of harmonic analysis, which consists in joining the given points by arcs of parabolas of the second or third degree and then obtaining the *exact* harmonic analysis of the artificial function defined by this series of arcs from the discontinuities.

Chapter VI, on theoretical considerations, I have added for the sake of completeness, and also because I find that most students of physics and engineering do show a considerable interest in the proofs of the various mathematical methods and operations taught them. Very few, in the course of their education, simply want to be given mathematical results that they can use and betray no interest in the establishment of the results.

In Chapter VII I deal with Fourier's integral theorem, the chief application of which is in the science of Optics; but since it is the limiting case of Fourier's theorem when the period is made infinite, it could not logically be excluded, and it is, moreover, wanted for the next chapter.

Chapter VIII deals with the detection and determination of any periodic components in any non-periodic function of which observations have been made over some finite range of time. The method adopted has been to apply the theorem of Chapter VII to the artificial function defined to be equal to the observations over the region for which these extend, and to be zero for all values outside this region.

The detection of periodic components in phenomena not themselves strictly periodic is required in analysing many meteorological and astronomical phenomena.

Finally, a very brief historical survey of the subject has been added in Chapter IX.

I need hardly say that I shall be grateful to any reader who will be kind enough to point out any mistakes or misprints. I shall also be grateful for any suggestions, especially from experienced practical workers, for incorporation in a further edition of the book should one be called for.

A. E.

*The University,*
*Manchester.*
*May*, 1925.

# CONTENTS

## CHAPTER VI

# A PRACTICAL TREATISE ON FOURIER'S THEOREM AND HARMONIC ANALYSIS

## INTRODUCTION

### ON IMAGINARY QUANTITIES

Before proceeding with the proper subject-matter of this book, it is desirable, we think, to give a very brief account of the so-called, and very badly called, "imaginary quantities," as we shall frequently have to make use of them. It is almost impossible, in mathematical work, to deal with sines and cosines without imaginary quantities inevitably presenting themselves; so it is very important at the outset to get clear ideas about them and about their relations to the trigonometrical functions.

Let O and A represent any two points in a plane and let V represent the "vector" OA, *i.e.* V represents a step of *magnitude* OA in the *direction* OA and in the *sense* from O to A. This being so, it is evident that 2V will represent a step of magnitude equal to twice the distance from O to A in the direction from O to A; and that in general if $a$ is a positive quantity $a$V will represent a step of magnitude $a$ times OA in the same direction and sense. Further, $-$V will clearly, with the ordinary meaning of the minus sign, represent something which will cancel or neutralise V, and therefore represents a step of magnitude equal to OA

in the sense *from* A to O; and similarly $-aV$ represents a step of $a$ times OA in the direction and sense of AO. Hence the effect of the factor $(-1)$ is to reverse the direction of a vector without altering its magnitude. But reversing the direction of a vector is equivalent to rotating it through two right angles or $\pi$ radians. Similarly, $(-1)$ times $-V$ represents $-V$ rotated through $\pi$ radians or V rotated through $2\pi$ radians. Proceeding in this manner, we see that if $n$ is a positive integer $(-1)^n V$ represents V turned through an angle of $n\pi$ radians. Hence we may say briefly that the factor $(-1)^n$ *represents* a rotation of $n\pi$ radians.

Now, exactly as in elementary algebra when dealing with indices, we define them first for positive integral exponents and then find we must interpret them for fractional or negative values because our mathematical operations give rise to them, so we must do the same here. If $(-1)^{m+n}$ is *always* to represent a rotation of $(m+n)\pi$ radians whenever $m+n$ is an integer $(-1)^n$ *must* represent a rotation through $n\pi$ radians, whether $n$ is positive or negative, integral or fractional.

Hence we take $(-1)^n$ as representing a rotation of $n\pi$, to be universally true. Writing $n\pi = \theta$ or $n = \theta/\pi$, we see that the factor $(-1)^{\theta/\pi}$ applied to any vector rotates it through an angle $\theta$. In particular, the factor $(-1)^{\frac{1}{2}}$ or $\sqrt{-1}$, which is generally denoted by $i$, represents a rotation of $\dfrac{\pi}{2}$, *i.e.* rotation through a right angle. If we denote the vector V turned through an angle $\theta$ by $V_\theta$, we have the result that, in general,

$$V_\theta = (-1)^{\theta/\pi} V,$$

and in particular,

$$V_{\frac{\pi}{2}} = i V.$$

We will now obtain another expression for $V_\theta$ in terms of V. Let OA in Fig. 1 denote the vector V and OB denote $V_\theta$. Draw BC perpendicular to OA, then OC denotes the vector V cos $\theta$ or (cos $\theta$) . V. Similarly, CB is a vector whose magnitude is equal to that of (sin $\theta$) . V, but since the vector CB has been rotated through a right angle away from the direction of V it is represented *as a vector* by $i$ sin $\theta$ . V. But the vector OB is equal to the vector OC plus the vector

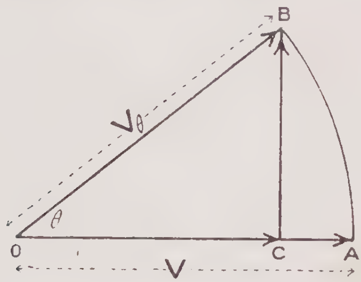


FIG. 1.

CB, which is merely asserting that the step OB is equal to the step OC plus the step CB. Hence we have

$$V_\theta = \cos\theta \,.\, V + i \sin\theta \,.\, V$$

or $$V_\theta = (\cos\theta + i\sin\theta) \,.\, V.$$

But we had previously found that

$$V_\theta = (-1)^{\theta/\pi}\, V,$$

hence we see that the two expressions $(-1)^{\theta/\pi}$ and $\cos\theta + i\sin\theta$ are equal and both represent rotation through an angle $\theta$. If $n$ is any integer and we replace $\theta$ by $n\theta$, we have

$$(-1)^{\frac{n\theta}{\pi}} = \cos n\theta + i\sin n\theta,$$

but $$(-1)^{\frac{n\theta}{\pi}} = \{(-1)^{\theta/\pi}\}^n = \{\cos\theta + i\sin\theta\}^n.$$

We thus arrive at the result which is known in trigonometry as De Moivre's Theorem, viz. that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

De Moivre's Theorem is generally regarded as belonging to the advanced part of trigonometry, yet, as we have seen, it is nothing more than the expression of the fact that a rotation of $n\theta$ is equal to $n$ rotations of $\theta$.

We can, by De Moivre's theorem, easily obtain the power series for the sine and cosine. If we write $\theta = x/n$ we get

$$\cos x + i\sin x = \left(\cos\frac{x}{n} + i\sin\frac{x}{n}\right)^n.$$

Now let $n$ tend to infinity, while $x$ is finite; so that $\cos\frac{x}{n} + i\sin\frac{x}{n}$ represents a rotation through an infinitesimal angle $\frac{x}{n}$, and thus reduces to $1 + i\frac{x}{n}$ in the limit. We then have

$$\cos x + i\sin x = \left(1 + i\frac{x}{n}\right)^n \quad . \quad . \quad (1)$$

when $n$ is infinite. Expanding this by the Binomial Theorem, we obtain

$$\cos x + i\sin x = 1 + ix + \frac{\left(1-\frac{1}{n}\right)i^2x^2}{\lfloor 2} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)i^3x^3}{\lfloor 3} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)i^4x^4}{\lfloor 4} + .$$

when $n$ is infinite.

If we remember that $i^2 = -1$; $i^3 = -i$, etc., and that, when this is done, the terms devoid of the factor $i$ represent displacements in the line of V; and that

those which contain the factor $i$ represent displacements at right angles to V, we see that

$$\cos x = 1 - \left(1 - \frac{1}{n}\right)\frac{x^2}{\lfloor 2} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right)\frac{x^4}{\lfloor 4} -$$

and

$$\sin x = x - \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{x^3}{\lfloor 3} + \ . \ . \ .$$

when $n$ is infinite.

Now both these series, when $n$ is very large, are such that for any given value of $x$ a *finite* number of terms will give their sums to any desired degree of accuracy. Hence we may safely put $n = \infty$ or $1/n = 0$ in these series. For in any term of finite order there are only a finite number of the factors $\left(1 - \frac{1}{n}\right)$ . . . $\left(1 - \frac{2}{n}\right)$, etc., and these give *exactly* unity when $1/n$ is *exactly* zero. Hence we have the important results that

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \ . \ . \ . \quad (2)$$

and

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \ . \ . \ . \quad (3)$$

It is shown in books on Algebra that the limit, when $n$ is infinite, of the expression

$$\left(1 + \frac{z}{n}\right)^n$$

is $e^z$, where $e = 2{\cdot}71821$ . . . and is the base of the Naperian logarithms, hence we see by (1) that

$$\cos x + i \sin x = e^{ix} \quad . \ . \ . \quad (4)$$

This is consistent with the previous value, viz. $(-1)^{x/\pi}$, found for the left-hand side since it only

implies that $(-1) = e^{i\pi} \equiv \cos \pi + i \sin \pi$. If we change $\theta$ into $-\theta$ in the equation $\cos \theta + i \sin \theta = e^{i\theta}$, we get

$$\cos \theta - i \sin \theta = e^{-i\theta},$$

adding and subtracting these equations we have

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

These results are known as the exponential values of the sine and cosine. Their meaning can easily be

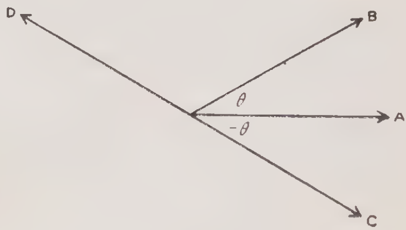


FIG. 2.

grasped from the diagram, Fig. 2. The factor $e^{i\theta}$, since it is the same as $\cos \theta + i \sin \theta$, represents a rotation through an angle $\theta$. Hence in Fig. 2 OA, OB and OC represent respectively V, $e^{i\theta}$ V and $e^{-i\theta}$ V. Clearly the sum of the latter two vectors is a vector equal to 2 $\cos \theta$ times V. Similarly OD represents the vector $-e^{-i\theta}$ V and the resultant of OB and OD is clearly a vector of magnitude equal to $2 \sin \theta$ V, but at right angles to it, and hence represented by $2 i \sin \theta$ V; this is equivalent to the equation $e^{i\theta} - e^{-i\theta} = 2i \sin \theta$.

Any expression of the form $a + bi$ is called a Complex Quantity; $a$ is said to be its "real" part and $bi$, or briefly $b$, is said to be its "imaginary" part. Much

better and more appropriate names would be "axial" part and "non-axial" part. From what has been said above it is clear that $(a + bi)$ V represents a vector with a component $a$V along V and a component $b$V at right angles to V. It thus represents a vector equal in magnitude to $\sqrt{a^2 + b^2}$ V, and making an angle $\tan^{-1} b/a$ with V. But this vector can be represented by $e^{i\theta}\sqrt{a^2 + b^2}$ V, where $\theta = \tan^{-1} b/a$.

Hence we see that

$$a + bi = re^{i\theta}; \text{ where } r = \sqrt{a^2 + b^2}$$

and $\theta = \tan^{-1} b/a$; a result that can be obtained immediately by writing $a + bi$ in the form

$$\sqrt{a^2 + b^2}\left\{\frac{a}{\sqrt{a^2 + b^2}} + \frac{bi}{\sqrt{a^2 + b^2}}\right\}.$$

The quantity $r$ is called the "*modulus*" of the complex quantity $a + bi$, and the quantity $\theta$ its "*argument*." The modulus of any quantity Q is often denoted by $|\,Q\,|$; thus if Q is a real negative quantity $|\,Q\,|$ denotes its numerical magnitude taken positively.

## Some Trigonometrical Theorems

It is convenient to place here some results which we shall require later so as to prevent the argument then from being interrupted by lengthy digressions.

The formulæ for the resolution of the product of two sine or cosine terms into the sum of two terms are so frequently required that we will first state them here for reference. They are

$$\cos A \cos B = \tfrac{1}{2}\cos(A + B) + \tfrac{1}{2}\cos(A - B),$$
$$\sin A \sin B = \tfrac{1}{2}\cos(A - B) - \tfrac{1}{2}\cos(A + B)$$

and

$$\sin A \cos B = \tfrac{1}{2}\sin(A + B) + \tfrac{1}{2}\sin(A - B).$$

If $n$ is any integer we have the well-known results that

$$\int_0^{2\pi} \cos nx\,dx = 0 \;.\;.\;.\; \text{(A)} \quad \text{and} \quad \int_0^{2\pi} \sin nx\,dx = 0 \;.\;.\;.\; \text{(A)}$$

For the indefinite integrals are respectively $\frac{\sin nx}{n}$ and $-\frac{\cos nx}{n}$, and both these expressions take the same values when $x = 2\pi$ as when $x = 0$. From these results we can easily deduce that, if $m$ is also an integer,

$$\int_0^{2\pi} \cos mx \cos nx dx = 0 \text{ if } m \neq n \text{ and}$$
$$= \pi \text{ if } m = n \quad . \quad . \quad . \quad . \quad \text{(B)}$$

$$\int_0^{2\pi} \sin mx \sin nx dx = 0 \text{ if } m \neq n \text{ and}$$
$$= \pi \text{ if } m = n \quad . \quad . \quad . \quad . \quad \text{(C)}$$

and $\int_0^{2\pi} \sin mx \cos nx dx = 0$ in all cases . . . (D)

To prove the first, we write $\cos mx \cos nx = \frac{1}{2} \cos (m + n)x + \frac{1}{2} \cos (m - n)x$; and, since $m + n$ and $m - n$ are both integers, the result follows at once from (A) unless $m = n$. In this case, the last term is $\frac{1}{2}$ and the integral of this from $0$ to $2\pi$ is $\pi$. The result (C) follows in an exactly similar manner. For (D) we write $\sin mx \cos nx = \frac{1}{2} \sin (m + n)x + \frac{1}{2} \sin (m - n)x$, from which it is seen by (A) that this is zero in all cases, even when $m = n$; for in this case the second term vanishes.

Some summation results, of which the integral results just given are the limiting case, will now be given. If $p$ is any integer we have

$$\cos \frac{2\pi}{p} + \cos \frac{4\pi}{p} + \cos \frac{6\pi}{p} + \ . \ . \ . + \cos \frac{2\pi p}{p} = 0$$

or as it is professionally written,

$$\sum_{q=1}^{p} \cos \frac{2\pi q}{p} = 0 \quad . \quad . \quad . \quad . \quad \text{(E)}$$

The $q$ under the $\Sigma$ indicates which letter the summation is with respect to, and the 1 below and the $p$ above indicate the limits of $q$. This result is obvious geometrically, for $\frac{2\pi}{p}$ is the external angle of a regular polygon of $p$ sides; so that the sum in question represents the sum of the projections of the sides, each of unit length, of a closed regular polygon on its base, which we have taken as the *zero*th or $p$th side.

Similarly,

$$\sum_{q=1}^{p} \sin \frac{2\pi q}{p} = 0 \quad . \quad . \quad . \quad . \quad \text{(F)}$$

for this is the projection of the same sides on a line at right angles to the base.

Further, if $n$ is any integer,

$$\sum_{q=1}^{p} \cos \frac{2\pi nq}{p} = \sum_{q=1}^{p} \sin \frac{2\pi nq}{p} = 0 \quad . \quad \text{(G)}$$

unless $n$ is a multiple of $p$, when the sum is $p$. For if $n$ is prime to $p$, the first sum still represents the sum of the projections of the sides of the same polygon on its base line, but taken in the order $n$th, $2n$th, $3n$th, etc., and in this sum each side is included once. If $n$ and $p$ have a common factor $r$, it can easily be seen that this sum will *not* include all the sides of the polygon, but that every $r$th side will be taken $r$ times. But the $p/r$ sides that are taken would, by themselves, form a closed regular polygon of $p/r$ sides, so that the sum of their projections is still zero. The only exceptional case is when $n$ is a multiple of $p$, then each term in the summation represents the projection of the base and therefore the sum is $p$ in the first case and zero in the second.

We can now prove that if $m$ and $n$ are integers *not greater than $p/2$* both

$$\sum_{q=1}^{p} \cos \frac{2\pi mq}{p} \cos \frac{2\pi nq}{p} = 0 \quad . \quad . \quad \text{(H)}$$

and
$$\sum_{q=1}^{p} \sin \frac{2\pi mq}{p} \sin \frac{2\pi nq}{p} = 0 \quad . \quad . \quad \text{(I)}$$

if $m \neq n$, while both sums are equal to $p/2$ if $m = n$, save in the exceptional case when $m = n = p/2$, when the sum in (H) is $p$ and the sum in (I) is zero.

These results can readily be obtained by replacing the products by the sum or difference of two cosines of the form $\cos \dfrac{2\pi(m \pm n)q}{p}$. The restriction on $m$ and $n$ is necessary to prevent $m + n$ becoming equal to $p$ or a multiple of it when $m \neq n$.

Similarly, we can prove that with the same limitations on $m$ and $n$

$$\sum_{q=1}^{p} \sin \frac{2\pi mq}{p} \cos \frac{2\pi nq}{p} = 0 \quad . \quad . \quad \text{(J)}$$

in all cases.

Lastly for this *Introduction* the definite integral $\displaystyle\int_0^\infty \frac{\sin bx}{x}\, dx$ is very important in the theory of Fourier's series. It may be evaluated as follows:

The indefinite integral $\displaystyle\int e^{cx} dx = \frac{e^{cx}}{c}$, of course, whether $c$ be real or complex. Let us write $c = -a + ib$, where $a$ and $b$ are positive, then we have,

$$\int_0^\infty e^{-ax}(\cos bx + i \sin bx)dx = \left[\frac{e^{(-a+ib)x}}{-a+ib}\right]_0^\infty = \frac{-1}{-a+ib}$$

$$= \frac{a+ib}{(a+ib)(a-ib)} = \frac{a+ib}{a^2+b^2}.$$

Equating real and imaginary parts we have,

$$\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2};$$

and

$$\int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}.$$

In the first result, the integrand is a "function" of $b$ and the given result holds for all values of $b$. Let us suppose, then, that we write down this result for an infinite number of values of $b$ between 0 and B each differing by an equal infinitesimal amount $db$ from the preceding value and add the whole set of equations after multiplying by $db$. The expression *under the integral sign* then becomes $\int_0^B e^{-ax} \cos bx \, db$, which $= \dfrac{e^{-ax} \sin Bx}{x}$, and the right-hand side becomes

$$\int_0^B \frac{a\,db}{a^2 + b^2}, \text{ which} = \tan^{-1} B/a.$$

Hence we have

$$\int_0^\infty \frac{e^{-ax} \sin Bx}{x} \, dx = \tan^{-1} B/a.$$

If now we make $a = 0$, the integral is still convergent and the result becomes

$$\int_0^\infty \frac{\sin Bx}{x} \, dx = \tan^{-1} \infty = \frac{\pi}{2}.$$

*Example.*—By integrating both sides of this result with respect to B from 0 to $2b$ obtain the result

$$\int_0^\infty \frac{\sin^2 bx}{x^2} \, dx = \frac{\pi}{2} b.$$

# CHAPTER I

## PERIODIC FUNCTIONS IN GENERAL

**Classes of Functions.** Mathematical functions may, for the purposes of the practical mathematician, be divided into three classes, viz. (*a*) What are known as "Analytic Functions" which we may perhaps call "Natural Functions" such as, $x^n$, $\sin x$, $e^x$, $\log x$, etc., and combinations of them. These functions exist, not only for all real values of $x$, but also for all complex values of $x$ of the form $x + iy$.*

(*b*) What we may call "Artificial Functions." These are functions which are arbitrarily defined as being equal to one analytic function in one part of their range and equal to another one in some other part. For instance, the function defined by

$$f(x) = 0 \text{ when } x < -1$$
$$f(x) = h \text{ when } -1 < x < +1$$
$$\text{and } f(x) = 0 \text{ when } x > +1$$

is an artificial function which is hereby defined for all real values of $x$.† Artificial functions must only be taken as existing in the regions for which they have been defined; and thus must never be taken as existing for complex values of $x$ unless they have been specifically defined for such complex values. (*c*) The third

* Some complicated analytic functions only exist over definite regions of the $xy$ plane, but this does not affect the argument.

† Strictly speaking, we have not defined the function for the values $x = \pm 1$. We could have got over this by saying $f(x) = h$ when $-1 \leq x \leq +1$, where $\leq$ means is "less than or equal to." For practical purposes, however, this is an unnecessary bit of hair splitting.

class are known as "Empirical Functions"; these are functions which we arrive at as the result of some experiments or observations where we obtain the result in the form of a graph or a set of points. In either case, the ideal function which we had in mind, and which it was the object of the experiment to determine, is only known to a limited accuracy owing to necessary experimental errors. We are supposed not to know the "natural" or "artificial" function which represents this ideal function, as otherwise we should,

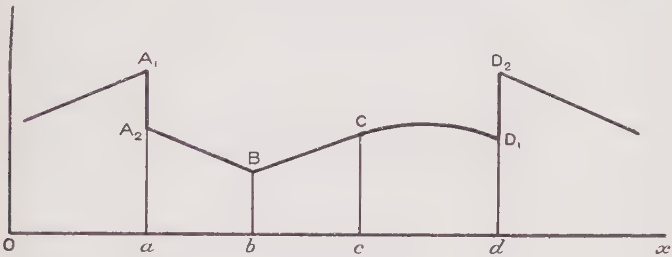


FIG. 3.

of course, use this representation, instead of the experimentally obtained values, in any mathematical operations we had to perform on it.

It is almost entirely to artificial and empirical functions to which we desire to apply Fourier's Theorem and the method of Harmonic Analysis.

**Discontinuities of Artificial Functions.** The most important distinction between an artificial function and an analytic one, is the possession by the former of what are called "discontinuities," the nature of which we must now briefly study. The heavy broken line in Fig. 3 represents an artificial function which possesses discontinuities at the points where $x = a$, $b$, $c$ or $d$. At $x = a$ there is an abrupt change in the magnitude of the function represented by $A_1 A_2$; at $x = b$, there is an abrupt change in slope but no change in magni-

tude, while at C there is an abrupt change in the rate of change of slope, or curvature, but no change either in magnitude or slope. Thus at any point of discontinuity of a function there is an abrupt change, either in the magnitude of the function, or of some of its differential coefficients. In general, if there is an abrupt change in the magnitude of any of these quantities, there is also an abrupt change in the magnitude of *all the higher differential coefficients* at the same points unless these are zero. It is accidental, so to speak, if a function changes in magnitude, but not in slope, as at $x = d$ in Fig. 3.

The discontinuities are measured by the magnitude of the abrupt changes at them. The ordinate of the point $A_1$ is denoted by $f(a - 0)$ and that of $A_2$ by $f(a + 0)$ and the magnitude of the change, $f(a + 0) - f(a - 0)$, we shall denote by $I_a$ so that $I_a$ stands for the magnitude of the abrupt change in $f(x)$ at $x = a$. Similarly the slope of the arc $A_2B$ at B is denoted by $f'(b - 0)$ and the slope of BC at B is denoted by $f'(b + 0)$, thus $f'(b + 0) - f'(b - 0)$ is the magnitude of the discontinuity of $f'(x)$ at $x = b$, and we shall denote it by $I'_b$. Similarly, the magnitude of the discontinuity of $f''(x)$ at $x = c$ will be denoted by $I''_c$ and so on.

**Periodic Functions.** In the great majority of practical cases the independent variable in these functions is either the time or an angular magnitude. It will thus be more convenient to denote it by $t$ than by $x$. A function $f(t)$ is said to be a periodic function when a constant $T$ exists, such that $f(T + t) = f(t)$ *for all values of t*. Changing $t$ into $T + t$ we have $f(2T + t) = f(T + t) = f(t)$; and in general $f(nT + t) = f(t)$ where $n$ is an integer. The least value of $T$ for which the equation $f(T + t) = f(t)$ is always satisfied is called *the period* of the function. Any multiple of *the period* is said to be *a period* of the

function. The behaviour of a periodic function over the range from $t = -\infty$ to $t = +\infty$ is thus a continual repetition of its behaviour over the range from 0 to $T$.

It is generally convenient, in discussing periodic functions, to change the scale of the $t$ axis so that the period is $2\pi$ as this will be found to make a great simplification in all the expressions. Actually, of course, in a diagram, we should mark the period divided into 360° and not into $2\pi$ radians. Sometimes it is advisable to take an aliquot part of $2\pi$ as the period. For instance, the length of the radius vector from the centre of an ellipse has an angular period of $\pi$, and it is better to keep it such than to change the variable. But we shall suppose in future that the period of a periodic function is always $2\pi$ unless otherwise expressed.

The student must grasp clearly the difference between a *natural* (or *analytic*) periodic function and an *artificial* one. If a periodic function of period $2\pi$ is represented by one and the same analytic expression over a whole range of $2\pi$, the function will nevertheless be an artificial periodic function, unless the *same analytic expression* also represents it for all values (to $\pm\infty$) outside this range. For instance the periodic function, of period $2\pi$, represented from $t = 0$ to $t = 2\pi$ by $e^{mt}$ is an artificial periodic function with discontinuities at $t = 0, \pm 2\pi, \pm 4\pi$, etc.: so too is a function of the same period and represented from $-\pi$ to $+\pi$ by $\sin pt$ *unless $p$ is an integer;* for it is only in this case that the periodic function we have defined is also represented *outside* the range from $-\pi$ to $+\pi$ by $\sin pt$. As another illustration, the function of period $2\pi$ defined by

$$f(t) = \frac{1}{2 + \cos t}$$

is an analytic periodic function; and so, it will easily

be seen, is the function defined by any rational expression which only involves sines or cosines of integral multiples of $t$.

**Four Special Classes of Periodic Functions.** Besides the general periodic functions, which have no symmetry, there are four important special classes

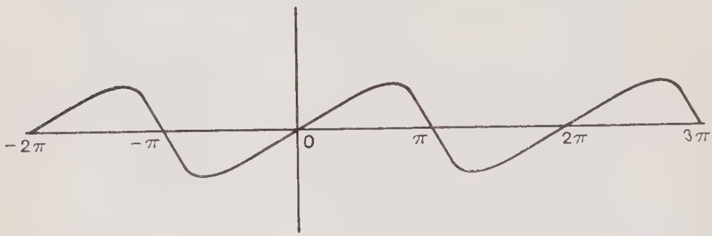


Fig. 4.

characterised by different degrees of symmetry which enables them to be easily distinguished from one another by the eye. We have :

($a$) Class I called "sine-harmonic functions." A periodic function belongs to this class when it is

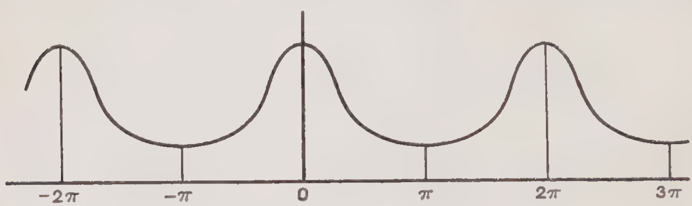


Fig. 5.

possible to *choose the origin* so that $f(t) = -f(-t)$ *for all values of t*. The function is then an *odd function.*

Writing $\pi + t$ for $t$ in the equation $f(t) = -f(-t)$, we have $f(\pi + t) = -f(-\pi - t), = -f(\pi - t)$, since $2\pi$ is a period, and hence the function is also an odd function when the origin is shifted *half* a period. An example of this class of periodic function is illustrated in Fig. 4.

(*b*) Class II, called "cosine-harmonic functions." A periodic function belongs to this class when it is possible to *choose the origin*, so that $f(t) = +f(-t)$ *for all values of t.* The function is then an even func-

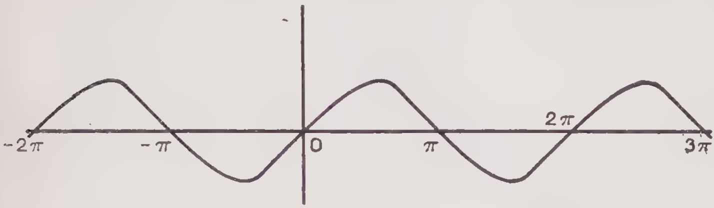


Fig. 6.

tion. Writing $\pi + t$ for $t$ in this equation, we have $f(\pi + t) = f(-\pi - t), = f(\pi - t)$, since $2\pi$ is a period; and hence the function is also an even function when the origin is shifted *half* a period. An example is illustrated in Fig. 5.

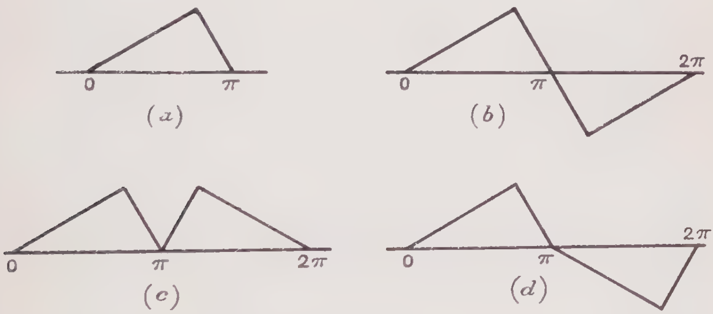


Fig. 7.

(*c*) Class III, called "odd-harmonic functions." A periodic function belongs to this class when $f(t + \pi) = -f(t)$ for all values of $t$, when $\pi$ is half the period. This relation is independent of the origin. An example is illustrated in Fig. 6.

These three classes of functions can be obtained by

repeating any given half wave in different manners, see Fig. 7, where (*a*) is supposed to represent a half period wave.

If the second half wave be obtained by rotating the first half through 180° about the point $\pi$, we obtain a Class I function as in Fig. 7 (*b*); while if it is repeated by reversing it fore and aft as in Fig. 7 (*c*) we have a Class II function. If the repetition is obtained by rotating the half wave about the axis of $t$, and then displacing it half a period along this axis, we have a Class III function, as in Fig. 7 (*d*). A mere repetition of the half wave, without any turning about at all,

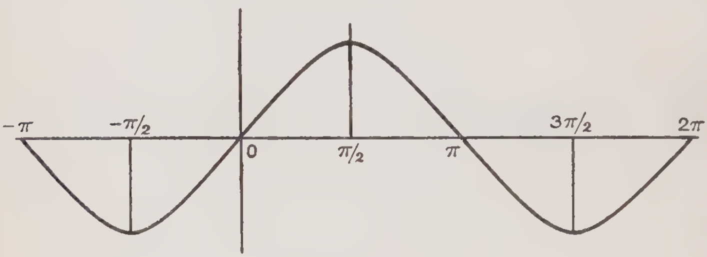


Fig. 8.

would, of course, result in the period being no longer $2\pi$, but $\pi$.

(*d*) Class IV, called "bi-symmetric functions." These are the periodic functions with the highest possible degree of symmetry. Each quarter period is geometrically similar to every other quarter period. An example is illustrated in Fig. 8. It will be seen that the functions of this class belong to all the three previous classes. With the origin as shown the function is an odd function, if the origin were shifted to $\frac{\pi}{2}$, the function would clearly be an even function; moreover, the function is also obviously a Class III function. The student should prove to himself that

any function that belongs to *any two* of the first three classes necessarily belongs to the other one also, and is a Class IV function.

It is important to be able to recognise to which of these four classes any periodic curve that occurs in any problem belongs. Let us consider, for instance, the open circuit voltage wave form of an alternator. Since in practice a "north" field pole is made geometrically similar to the adjacent "south" one, and since both are symmetrical about their centre lines, it is clear that the voltage wave form must be a Class IV function. This is not true, however, when the machine is on load, for then, as is well known, the field due to the armature current weakens that due to the field poles on the sides at which the armature is approaching the poles and strengthens it on the leaving sides. Nevertheless, it is clear that after half a period, the induced voltage will still be numerically equal, but of opposite sign, and thus that the voltage wave form is a Class III function.

*Example* 1.—Prove that any periodic function can be represented as the sum of an odd and an even periodic function.

We have $f(t) = \dfrac{f(t) - f(-t)}{2} + \dfrac{f(t) + f(-t)}{2}$ identically.

The first term merely changes sign on changing $t$ into $-t$ and the second one is unaltered thereby. Hence the first term represents a Class I function and the second a Class II one.

*Example* 2.—Prove that any periodic function can be represented as the sum of a Class III function and another of half the period.

We have $$f(t) = \frac{f(t) - f(t+\pi)}{2} + \frac{f(t) + f(t+\pi)}{2}.$$

The first term merely changes sign on changing $t$ into $t + \pi$, since $2\pi$ is the period. The second term is unaltered by this change and hence $\pi$ must be a period of this term.

*Example* 3.—Prove that any Class III function can be represented as the sum of two Class IV functions.

For if the function be represented as the sum of an odd and an

even function by Ex. 1 it will be seen that both these functions are still Class III functions and must hence be Class IV functions.

*Example* 4.—(*a*) Prove that $a_0 + a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t + \ldots$ represents a Class II function of period $2\pi$.

(*b*) Prove that $b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \ldots$ represents a Class I function of period $2\pi$.

(*c*) Prove that $a_1 \cos t + a_3 \cos 3t + a_5 \cos 5t + \ldots + b_1 \sin t + b_3 \sin 3t + b_5 \sin 5t + \ldots$ represents a Class III function of period $2\pi$.

Each of these series is supposed convergent.

*Example* 5.—Prove that either of the series

(*a*) $a_1 \cos t + a_3 \cos 3t + a_5 \cos 5t + \ldots$

and (*b*) $b_1 \sin t + b_3 \sin 3t + b_5 \sin 5t + \ldots$ represents a Class IV function of period $2\pi$.

For the first series represents a function belonging to Classes II and III and the second a function belonging to Classes I and III.

If the origin in the second series be shifted to the point $t = \frac{\pi}{2}$ by writing $t = \frac{\pi}{2} + t'$ the series becomes

$$b_1 \sin\left(\frac{\pi}{2} + t'\right) + b_3 \sin 3\left(\frac{\pi}{2} + t'\right) + b_5 \sin 5\left(\frac{\pi}{2} + t'\right) + \ldots$$

which

$$= b_1 \cos t' - b_3 \cos 3t' + b_5 \cos 5t' - \ldots$$

showing that both these series represent the same kind of functions referred to different origins.

*Example* 6.—Sketch the curves given by

(*a*) $f(t) = \cos t + \cdot 2 \cos 2t$.
(*b*) $f(t) = \sin t + \cdot 2 \sin 2t$.
(*c*) $f(t) = \sin t + \cdot 1 \sin 3t$.
(*d*) $f(t) = \sin t - \cdot 1 \sin 3t$.
(*e*) $f(t) = \sin t + \cdot 1 \cos 3t$.

and verify in each case that your sketch belongs to the correct class of function.

# CHAPTER II

## FOURIER'S THEOREM

**Statement of Theorem.** Fourier's Theorem states that any periodic function, whether "natural" or "artificial," of period $2\pi$ can be expressed by the series—

$$f(t) = \left.\begin{array}{l} \frac{a_0}{2}^{*} + a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t + \ldots \\ \quad + b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \ldots \end{array}\right\} \quad \ldots \quad (1)$$

where

$$\left.\begin{array}{l} a_n = \frac{1}{\pi}\int_0^{2\pi} f(t) \cos nt\, dt \\ \text{and} \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(t) \sin nt\, dt \end{array}\right\} \quad \ldots \quad (2)$$

There are a few unimportant restrictions on the behaviour of $f(t)$ which we will notice later in order that the theorem should be true.

We cannot at this stage *prove* Fourier's theorem, for this requires the proof that the representation of the function by the series given in (1) is possible; but we can easily prove that *if* $f(t)$ can be represented by the given series the coefficients must have the values given by (2). The two sides of (1), being supposed

* The reason for writing $\frac{a_0}{2}$ instead of $a_0$ is so that the value of $a_0$ will be correctly given by putting $n = 0$ in the expression for $a_n$.

identical for all values of $t$, must remain identical when the same operation is performed on both sides; unless this renders the series divergent. Let us multiply both sides of (1) by $dt$ and integrate from $t = 0$ to $t = 2\pi$. By equations (A), p. 7, all the terms on the right-hand side vanish except the first, so we are left with

$$\int_0^{2\pi} f(t)\,dt = \frac{a_0}{2}\int_0^{2\pi} dt = \pi a_0,$$

which is the value obtained by putting $n = 0$ in (2).

Again multiplying both sides of (1) by $\cos nt\,dt$ and integrating from 0 to $2\pi$ we see that all the terms on the right-hand side vanish by (B) and (D), p. 8, except the $a_n \cos nt$ term; thus we get

$$\int_0^{2\pi} f(t)\cos nt\,dt = a_n\int_0^{2\pi} \cos^2 nt\,dt$$

$$= \frac{a_n}{2}\int_0^{2\pi}(1 + \cos 2nt)dt = \pi a_n,$$

which is the value of $a_n$ given by (2). Similarly, we have

$$\int_0^{2\pi} f(t)\sin nt\,dt = b_n\int_0^{2\pi} \sin^2 nt\,dt$$

$$= \frac{b_n}{2}\int_0^{2\pi}(1 - \cos 2nt)dt = \pi b_n,$$

also in agreement with (2). Hence the theorem is proved on the assumption that the expansion is possible.

In the expansion (1) the various terms are called the "harmonics" of $f(t)$, the cosine terms being called the "cosine harmonics" and the sine terms the "sine harmonics." Thus $a_1 \cos t$ is called the "first cosine

harmonic" and $b_2 \sin 2t$ is called the "second sine harmonic." Both the $n$th sine harmonic and the $n$th cosine harmonic, of course, go through $n$ complete cycles in the period of the functions. The two $n$th harmonic terms $a_n \cos nt + b_n \sin nt$ can be combined together in the form

$$\sqrt{a_n{}^2 + b_n{}^2} \cos (nt - \phi),$$

where $\tan \phi = b_n/a_n$.

This expression is called the resultant or complete $n$th harmonic; $\sqrt{a_n{}^2 + b_n{}^2}$ is its amplitude and $-\phi$ its initial phase. The "first harmonic" is often called instead, the "fundamental harmonic." The $a$'s and $b$'s in (1) are often called the "Fourier constants" of $f(t)$.

Two or three proofs of Fourier's Theorem will be given later. The student familiar with the theory of linear differential equations with constant coefficients will be interested in the following. If $f(t)$ has a period $2\pi$ we have $f(t + 2\pi) = f(t)$. By the symbolical form of Taylor's theorem $f(t + 2\pi) = e^{2\pi D} f(t)$ where $D \equiv \frac{d}{dt}$; hence $(e^{2\pi D} - 1) \cdot f(t) = 0$, which is a differential equation for $f(t)$. We know that if $m$ is a root of $e^{2\pi D} - 1 = 0$, the solution consists of the sum of a number of terms of the form $Ce^{mt}$. For two imaginary roots, $\pm im$, the terms take the form $A \cos mt + B \sin mt$, $A$, $B$ and $C$ being arbitrary constants. Now $e^{2\pi D} - 1 = 0$ has obviously no real root save $D = 0$; to search for imaginary roots we put $D = i\lambda$ and get $e^{2\pi i\lambda} = 1$ or $\cos 2\pi\lambda = 1$ and $\sin 2\pi\lambda = 0$ simultaneously; from which it is obvious that $\lambda$ can be any integer. Hence we see that $f(t)$ can be expressed in the form

$$f(t) = C + A_1 \cos t + A_2 \cos 2t + A_3 \cos 3t + \ldots$$
$$+ B_1 \sin t + B_2 \sin 2t + B_3 \sin 3t + \ldots$$

The constants must be chosen to satisfy the initial conditions, which in this case consist of the behaviour of the function from 0 to $2\pi$.

Ce

**Simplification of Fourier's Theorem for the Four Special Classes of Functions.** If $f(t)$ is an odd function, so that $f(t) = -f(-t)$ for all values of $t$, we have

$$\frac{a_0}{2} + a_1 \cos t + a_1 \cos 2t + \quad . \; . \; . + b_1 \sin t + \quad . \; . \; .$$
$$= - \left\{ \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + . \; . \; . \right.$$
$$\left. - b_1 \sin t - b_2 \sin 2t - . \; . \; . \right\}$$

since $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$.
This reduces to

$$a_0 + 2a_1 \cos t + 2a_2 \cos 2t + \quad . \; . \; . \quad = 0$$

for all values of $t$; hence all the $a$'s are zero, so that any odd periodic function can be expanded in the series

$$f(t) = b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \quad . \quad . \quad (3)$$

Similarly, if $f(t) = + f(-t)$ for all values of $t$, so that $f(t)$ is an even function, we find that all the $b$'s are zero and that

$$f(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + \quad . \quad . \quad (4)$$

Again, if $f(t)$ satisfies the condition that $f(t + \pi) = - f(t)$ for all values of $t$, we have

$$\frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + \quad . \; . \; .$$
$$+ b_1 \sin t + b_2 \sin 2t + \quad . \; . \; .$$
$$= - \frac{a_0}{2} - a_1 \cos(t + \pi) - a_2 \cos 2(t + \pi) -$$
$$a_3 \cos 3(t + \pi) \quad . \; . \; .$$
$$- b_1 \sin (t + \pi) - b_2 \sin 2(t + \pi) - b_3 \sin 3(t + \pi) \; . \; . \; .$$
$$= - \frac{a_0}{2} + a_1 \cos t - a_2 \cos 2t + a_3 \cos 3t - \; . \; . \; .$$
$$+ b_1 \sin t - b_2 \sin 2t + b_3 \sin 3t - \; . \; . \; . \; .$$

and hence

$$a_0 + 2a_2 \cos 2t + 2a_4 \cos 4t + 2a_6 \cos 6t + \quad . \; . \; .$$
$$+ 2b_2 \sin 2t + 2b_4 \sin 4t + 2b_6 \sin 6t + \quad . \; . \; .$$

$= 0$ for all values of $t$, whence

$a_0 = a_2 = a_4 = a_6 \ldots = 0$ and $b_2 = b_4 = b_6 \ldots = 0$, and the function can therefore be represented by

$$f(t) = a_1 \cos t + a_3 \cos 3t + \ldots$$
$$+ b_1 \sin t + b_3 \sin 3t + \ldots \quad (5)$$

It will be seen that these three cases correspond to Class I, Class II and Class III functions respectively, and the reason for the names "sine-harmonic," "cosine harmonic" and "odd-harmonic" will now be evident. We will now give simplified expressions for the Fourier constants in these cases. The expressions previously given for the Fourier constants, viz.:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t) \cos nt\, dt \text{ and } b_n = \frac{1}{\pi}\int_0^{2\pi} f(t) \sin nt\, dt$$

can be written

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt\, dt \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt\, dt\,;$$

for, since the expressions under the integral sign have $2\pi$ as a period, the result will be the same so long as the range of integration is $2\pi$, independently of the position from which the range commences.

For a Class I function with the origin suitably chosen, $f(t)$ is an odd function and since $\cos nt$ is even, the product $f(t) \cos nt$ is an odd function; hence the integral of it from $-\pi$ to $+\pi$ is zero. This also shows that in this case all the cosine terms vanish. Again, since $\sin nt$ is an odd function and the product $f(t) \sin nt$ is an even function, its integral from $-\pi$ to $+\pi$ is twice its integral from 0 to $\pi$.

Hence the result that a Class I function of period $2\pi$ may be represented by

$$f(t) = b_1 \sin t + b_2 \sin 2t + \ldots \quad (6)$$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t) \sin nt\, dt \quad \ldots \quad (7)$$

Similarly for a Class II function with the origin chosen so that it is also an even function; $b_n$ vanishes since $f(t)\sin nt$ is an odd function, while $a_n$ is given by double the integral over half the range.

Hence the result that a Class II function of period $2\pi$ may be represented by

$$f(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + \quad . \quad . \qquad (8)$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t) \cos nt\, dt \quad . \quad . \quad . \qquad (9)$$

Consider next a Class III function and break the integrals broken into two taken over the ranges $-\pi$ to 0 and 0 to $\pi$ respectively. If we consider corresponding elements in these two sections at a distance of $\pi$ apart we see that $f(t)$ has equal numerical values but opposite signs in the two cases, since $f(t+\pi) = -f(t)$. Further, since $\pi$ is equal to an integral number of periods of the even harmonic terms, the product $f(t)\cos nt$ or $f(t)\sin nt$ will, when $n$ is even, have equal numerical values but opposite signs at all corresponding elements and hence the integral from $-\pi$ to 0 will cancel that from 0 to $\pi$. If, however, $n$ is odd $\cos nt$ and $\sin nt$ will have an odd number of half wave-lengths and simply change sign in the range of $\pi$, so that $f(t)\cos nt$ and $f(t)\sin nt$ will have the same values at all corresponding elements and we may therefore take the integrals from 0 to $\pi$ and double them.

Hence the result that a Class III function of period $2\pi$ may be represented by

$$\left.\begin{aligned} f(t) &= a_1 \cos t + a_3 \cos 3t + a_5 \cos 5t + \quad . \quad . \quad . \\ &\quad + b_1 \sin t + b_3 \sin 3t + b_5 \sin 5t + \quad . \quad . \quad . \end{aligned}\right\} \qquad (10)$$

where

$$\left.\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt \\ \text{and} \qquad b_n &= \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt \end{aligned}\right\} \quad . \quad . \quad . \quad (11)$$

Lastly, let $f(t)$ be a Class IV function with the origin chosen so that it is an odd function. Then since it is also a Class III function we have

$$f(t) = b_1 \sin t + b_3 \sin 3t + \quad . \quad . \quad . \quad (12)$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt.$$

If $f(t)$ is referred to a new origin at $t = \frac{\pi}{2}$, it becomes an even function with respect to the new origin; so too does $\sin nt$ when $n$ is odd, for if $n = 2m + 1$ it becomes $\sin\ (2m + 1)\left(\frac{\pi}{2} + t'\right)$ which $= \cos\ \{m\pi + (2m + 1)t'\} = (-1)^m \cos\ (2m + 1)t'$, hence the given integral from 0 to $\pi$ is equal to twice the integral from 0 to $\frac{\pi}{2}$.

Hence the result that an odd Class IV function of period $2\pi$ can be represented by

$$f(t) = b_1 \sin t + b_3 \sin 3t + \quad . \quad . \quad . \quad (12)$$

where

$$b_n = \frac{4}{\pi}\int_0^{\pi/2} f(t)\sin nt\,dt \quad . \quad . \quad . \quad (13)$$

In an exactly similar manner it can be proved that an even Class IV function of period $2\pi$ can be represented by

$$f(t) = a_1 \cos t + a_3 \cos 3t + a_5 \cos 5t + \ . \quad . \quad (14)$$

where

$$a_n = \frac{4}{\pi}\int_0^{\pi/2} f(t) \cos nt\, dt \quad . \quad . \quad . \quad (15)$$

The student should guard against the mistake of working out these latter integrals *when* n *is even* and then *because the result is not zero* putting in even harmonic terms in the expansion of $f(t)$. The only reason why the range of integration is halved is because the values of $a_n$ and $b_n$ *are zero when* n *is even* and can be obtained by doubling the integral over half the range when $n$ is odd.

In functions of Classes I, II or IV, only one kind of integral has to be evaluated, viz. $\int f(t) \cos nt\, dt$ or $\int f(t) \sin nt\, dt$; but in the Class III functions, and in the case of the general function, both types are required. In these cases it is most convenient to multiply the expression for $b_n$ by $i$ and add it to the expression for $a_n$, so that

$$a_n + ib_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)[\cos nt + i \sin nt]dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)e^{int}\, dt;$$

and then after evaluating this integral to separate it into its real and imaginary parts which are equated to $a_n$ and $b_n$ respectively.

If the period of $f(t)$ is not $2\pi$ but an aliquot part of it, say $\frac{2\pi}{m}$, the formulæ we have given for the coefficients still hold, since $2\pi$ is still *a* period of the function. In this case all the harmonics vanish except those which are multiples of the $m$th,* so that

* Actually, of course, what appears here as the $m$th harmonic is physically the first or fundamental harmonic.

$$f(t) = \frac{a_0}{2} + a_1 \cos mt + a_2 \cos 2mt + \ldots$$
$$+ b_1 \sin mt + b_2 \sin 2mt + \ldots \quad (16)$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t) \cos mnt \, dt;$$

but since $f(t)$ and $\cos mnt$ both have a period of $\frac{2\pi}{m}$ this may be written

$$a_n = \frac{m}{\pi}\int_0^{\frac{2\pi}{m}} f(t) \cos mnt \, dt \quad . \quad . \quad . \quad (17)$$

and similarly for $b_n$.

*Example.*—Prove that if the period of $f(t)$ be T, Fourier's expansion becomes

$$f(t) = \frac{a_0}{2} + a_1 \cos \frac{2\pi t}{T} + a_2 \cos \frac{4\pi t}{T} + \ldots$$
$$+ b_1 \sin \frac{2\pi t}{T} + b_2 \sin \frac{4\pi t}{T} + b_3 \sin \frac{6\pi t}{T} + \ldots \quad (18)$$

where $$a_n = \frac{2}{T}\int_0^T f(t) \cos \frac{2\pi nt}{T} \, dt \quad . \quad . \quad . \quad . \quad (19)$$

and $$b_n = \frac{2}{T}\int_0^T f(t) \sin \frac{2\pi nt}{T} \, dt \quad . \quad . \quad . \quad . \quad (20)$$

We may conveniently close this chapter with

## Some Applications to Electrical Engineering.

**Isolation of Particular Classes of Harmonics in any Periodic Function.** If $f(t) = \frac{a_0}{2} + a_1 \cos t + \ldots + b_1 \sin t + \ldots$, and if we displace it by a distance $\pi$, the displaced function, $f(t + \pi)$, is given by

$$f(t+\pi)=\frac{a_0}{2}+a_1\cos(t+\pi)+\ldots+b_1\sin(t+\pi)+\ldots$$
$$=\frac{a_0}{2}-a_1\cos t+a_2\cos 2t-a_3\cos 3t+\ldots$$
$$-b_1\sin t+b_2\sin 2t-b_3\sin 3t+\quad\ldots$$

If we add this to the original function we get

$$f(t)+f(t+\pi)=2\left\{\frac{a_0}{2}+a_2\cos 2t+a_4\cos 4t+\ldots\right.$$
$$\left.+b_2\sin 2t+b_4\sin 4t+\ldots\right\}$$

that is, twice the sum of all the even harmonics. More generally, if we displace the function by the distances $\frac{2\pi}{p}, \frac{4\pi}{p}, \frac{6\pi}{p} \ldots 2\pi\frac{(p-1)}{p}$ in turn and add these $p-1$ displaced functions to the original, we get, since the sum of all the sine and cosine terms which are not multiples of the $p$th vanish by (G), p. 9,

$$\text{the Sum}=p\left\{\frac{a_0}{2}+a_p\cos pt+a_{2p}\cos 2\,pt+\ \ldots\right.$$
$$\left.+b_p\sin pt+b_{2p}\sin 2pt\ +\ldots\right\} \quad \ldots . \ (21)$$

that is, we get $p$ times the sum of all the harmonics which are multiples of the $p$th. Again, if we subtract $f\left(t+\frac{2\pi}{p}\right)$ from $f(t)$ it will be seen that all harmonics which *are* multiples of the $p$th will cancel out. As an example, let us find

**(1) The Terminal Voltage of a Three-Phase Generator.** Let $f(t)$ be the voltage induced in one phase of a three-phase star connected alternator. Then the voltages induced in the other

phases are given by $f\left(t + \frac{2\pi}{3}\right)$ and $f\left(t + \frac{4\pi}{3}\right)$, the latter of which may be equally well represented by $f\left(t - \frac{2\pi}{3}\right)$.

The voltage *between* the second and third phases is $f\left(t + \frac{2\pi}{3}\right) - f\left(t - \frac{2\pi}{3}\right)$. Since $f(t)$ is necessarily a Class III function (p. 19) we have

$$f(t) = a_1 \cos t + a_3 \cos 3t + a_5 \cos 5t + \dots$$
$$+ b_1 \sin t + b_3 \sin 3t + \dots,$$

and hence

$$f\left(t + \frac{2\pi}{3}\right) - f\left(t - \frac{2\pi}{3}\right)$$
$$= a_1\left\{\cos\left(t + \frac{2\pi}{3}\right) - \cos\left(t - \frac{2\pi}{3}\right)\right\} + \dots$$
$$+ b_1\left\{\sin\left(t + \frac{2\pi}{3}\right) - \sin\left(t - \frac{2\pi}{3}\right)\right\} + \dots$$

which easily reduces to

$$\sqrt{3}\{- a_1 \sin t + a_5 \sin 5t - a_7 \sin 7t + a_{11} \sin 11t \,..\}$$
$$+ \sqrt{3}\{b_1 \cos t - b_5 \cos 5t + b_7 \cos 7t - b_{11} \cos 11\, t + ..\}.$$

If in this we write $\sin t = -\cos\left(t + \frac{\pi}{2}\right)$ and $\cos t = \sin\left(t + \frac{\pi}{2}\right)$, etc., we get

$$\sqrt{3}\left\{a_1 \cos\left(t+\frac{\pi}{2}\right) - a_5 \cos 5\left(t+\frac{\pi}{2}\right) + a_7 \cos 7\left(t+\frac{\pi}{2}\right) - .\right.$$
$$\left. + b_1 \sin\left(t+\frac{\pi}{2}\right) - b_5 \sin 5\left(t+\frac{\pi}{2}\right) + b_7 \sin 7\left(t+\frac{\pi}{2}\right) - ..\right\}.$$

If we compare this with $f(t)$ itself we see that all the harmonics which are multiples of 3 have disappeared while the amplitudes of all the others are multiplied by $\sqrt{3}$. We note also that the coefficients of the 5th, 11th, 17th, etc., harmonics have their signs changed, which may be interpreted as a change in phase of the harmonic in question amounting to half its period.

**(2) Voltage Wave Form of an Alternator of given Field Form.** By the "field form" of an alternator we mean the voltage wave form induced in a single conductor. This is not in general the same as the voltage wave form at the terminals of the machine, since the armature generally contains a large number of conductors connected in series which occupy a displaced position relatively to one another on the armature. We will suppose that the windings are *uniformly spread* over an angle of $2\alpha$ radians, or of $2\alpha$ "electrical radians" if the machine has more than two field poles. If

$$\begin{aligned} f(t) = a_1 \cos t + a_3 \cos 3t + \ldots \\ + b_1 \sin t + b_3 \sin 3t + \ldots \end{aligned}$$

represents the voltage induced in the central conductor, then

$$\begin{aligned} f(t+\theta) = a_1 \cos (t+\theta) + a_3 \cos 3(t+\theta) + \ldots \\ + b_1 \sin (t+\theta) + b_3 \sin 3(t+\theta) + \ldots \end{aligned}$$

will represent the voltage induced in a conductor at an angular distance $\theta$ in advance of the central one, and the total induced voltage in all the conductors will be *proportional* to

$$\int_{-\alpha}^{\alpha} f(t+\theta)d\theta,$$

that is, to

$$a_1 \int_{-\alpha}^{\alpha} \cos (t + \theta) d\theta + a_3 \int_{-\alpha}^{\alpha} \cos 3(t + \theta) d\theta + \ldots$$

$$+ b_1 \int_{-\alpha}^{\alpha} \sin (t + \theta) d\theta + \ldots .$$

This is equal to

$$a_1\{\sin (t + \alpha) - \sin (t - \alpha)\} + \frac{a_3}{3}\{\sin 3(t + \alpha) - \sin 3(t - \alpha)\} + b_1\{\cos (t - \alpha) - \cos (t + \alpha)\} + \ldots,$$

or to

$$2a_1 \sin \alpha \cos t + \frac{2a_3}{3} \sin 3\alpha \cos 3t + \ldots$$

$$+ 2b_1 \sin \alpha \sin t + \frac{2b_3}{3} \sin 3\alpha \sin 3t + \ldots .$$

If we divide this expression by $2 \sin \alpha$, so as to make the amplitude of the first harmonic equal to its amplitude in the field form, we see that the amplitude of the $n$th harmonic in the voltage wave form is reduced (relatively) to $\dfrac{\sin n\alpha}{n \sin \alpha}$ of its value in the field form.

If, for instance, $\alpha = \pi/3$ or $60°$ all harmonics that are multiples of 3 will vanish and the field form

$$\begin{aligned} f(t) = a_1 \cos t + a_3 \cos 3t + \ldots \\ + b_1 \sin t + b_3 \sin 3t + \ldots \end{aligned}$$

will produce the voltage form

$$a_1 \cos t - \frac{a_5}{5} \cos 5t + \frac{a_7}{7} \cos 7t - \ldots$$

$$+ b_1 \sin t - \frac{b_5}{5} \sin 5t + \frac{b_7}{7} \sin 7t - \ldots .$$

D

*Example.*—If $\alpha = \frac{2\pi}{5}$ or $72°$ so that the spread of the windings is $144°$, prove that the voltage wave form of a star connected three-phase alternator will be proportional to

$$a_1 \cos t + \cdot 089 a_7 \cos 7t + \cdot 091 a_{11} \cos 11t - \cdot 048 a_{13} \cos 13t + .$$
$$+ b_1 \sin t + \cdot 089 b_7 \sin 7t + \cdot 091 b_{11} \sin 11t - \cdot 048 b_{13} \sin 13t + .$$

where

$$a_1 \cos t + a_3 \cos 3t + a_5 \cos 5t + \ldots$$
$$+ b_1 \sin t + b_3 \sin 3t + b_5 \sin 5t + \ldots$$

represents the field form.

[There is a difference of phase between these two expressions as written, but this does not affect the amplitudes of the various harmonics.]

**(3) The Root Mean Square (R.M.S.) Value of a Periodic Curve.** We frequently, especially in electrical engineering, require to know the square root of the mean square of the value of a periodic function; $f(t)$ being a periodic function of period $2\pi$, this quantity is clearly by definition,

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} f(t)^2\, dt\right\}^{\frac{1}{2}}.$$

Let us multiply both sides of the equation

$$f(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + \ldots$$
$$+ b_1 \sin t + b_2 \sin 2t + \ldots$$

by $f(t)\, dt$ and integrate from 0 to $2\pi$.

Since $$\int_0^{2\pi} f(t) \cos nt\, dt = \pi a_n$$

and $$\int_0^{2\pi} f(t) \sin nt\, dt = \pi b_n$$

we get

$$\int_0^{2\pi} f(t)^2\, dt = \frac{a_0}{2}(\pi a_0) + a_1(\pi a_1) + a_2(\pi a_2) + \ldots$$
$$+ b_1(\pi b_1) + b_2(\pi b_2) + \ldots;$$

and hence

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)^2\,dt = \frac{a_0^2}{4} + \frac{a_1^2 + b_1^2}{2} + \frac{a_2^2 + b_2^2}{2} + \ldots .$$

If we denote the constant term $\frac{a_0}{2}$ by $c_0$ and the amplitudes of the 1st, 2nd, 3rd . . . harmonics by $c_1, c_2, c_3 \ldots$, we see that the R.M.S. value of a periodic function is given by

$$\sqrt{c_0^2 + \frac{c_1^2 + c_2^2 + c_3^2 + \ldots}{2}}.$$

**(4) The Form Factor.** Another characteristic of a periodic function which is required in electrical engineering is the "form factor" of a voltage wave. This is defined as the ratio of the R.M.S. value to the *mean value* of the rectified wave.

Since the voltage wave of an ordinary alternator can only involve odd harmonics, we limit ourselves to that case. Let the voltage be given by

$$\begin{aligned} V = {} & b_1 \sin t + b_3 \sin 3t + \ldots \\ & + a_1 \cos t + a_3 \cos 3t + \ldots \end{aligned}$$

so that its R.M.S. value is

$$\sqrt{\frac{(a_1^2 + b_1^2) + (a_3^2 + b_3^2) + \ldots}{2}}.$$

We will suppose that $V = 0$ when $t = 0$ and that $b_1$ is positive so that the positive half of the wave extends from $t = 0$ to $t = \pi$. This of course necessitates the relation $a_1 + a_3 + a_5 + \ldots = 0$ among the cosine coefficients.

The mean value of the positive half of the wave is $\frac{1}{\pi}\int_0^{\pi} V\,dt$, which is readily seen to be equal to

$$\frac{2}{\pi}\left(b_1 + \frac{b_3}{3} + \frac{b_5}{5} + \ldots\right).$$

So that the form factor is

$$\frac{\pi}{2\sqrt{2}} \cdot \frac{\sqrt{(a_1^2 + b_1^2) + (a_3^2 + b_3^2) + \ldots}}{b_1 + b_3/3 + b_5/5 + \ldots}.$$

The numerical factor is equal to 1·111 approx.

Whenever the equation of a periodic function is known, both its R.M.S. value and its Form Factor can be deduced directly from the integrals defining them: it is quite unnecessary then to use the expressions for these quantities given above in terms of the Fourier constants.

# CHAPTER III

## ON THE REPRESENTATION OF ARTIFICIAL FUNCTIONS BY FOURIER'S SERIES

We shall in this chapter obtain the Fourier Series representing a variety of fairly simple artificial functions. The Fourier series representing a function is called briefly the "Harmonic Analysis" of the function.

*Problem I.*—Obtain the harmonic analysis of the even function represented by $f(t) = h$ from $t = 0$ to $t = \alpha$ and by $f(t) = 0$ from $t = \alpha$ to $t = \pi$.*

Since the function is even, we have

$$f(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + \quad . \quad . \qquad (1)$$

where $$a_n = \frac{2}{\pi}\int_0^{\pi} f(t) \cos nt \, dt.$$

If we substitute the given expressions for $f(t)$ over their appropriate ranges we get

$$a_n = \frac{2}{\pi}\int_0^{\alpha} h \cos nt \, dt,$$

which gives $$a_n = \frac{2h}{\pi n} \sin n\alpha.$$

* It will be assumed in all cases that the period is $2\pi$ unless otherwise stated.

This readily gives all the $a$'s save $a_0$ which takes an indeterminate form of $\frac{0}{0}$. If, however, we imagine $n$ to be *very small* but not quite zero, we see that the expression tends to $\frac{2h\,n\alpha}{\pi n} = \frac{2h\alpha}{\pi}$, when $n$ is zero.

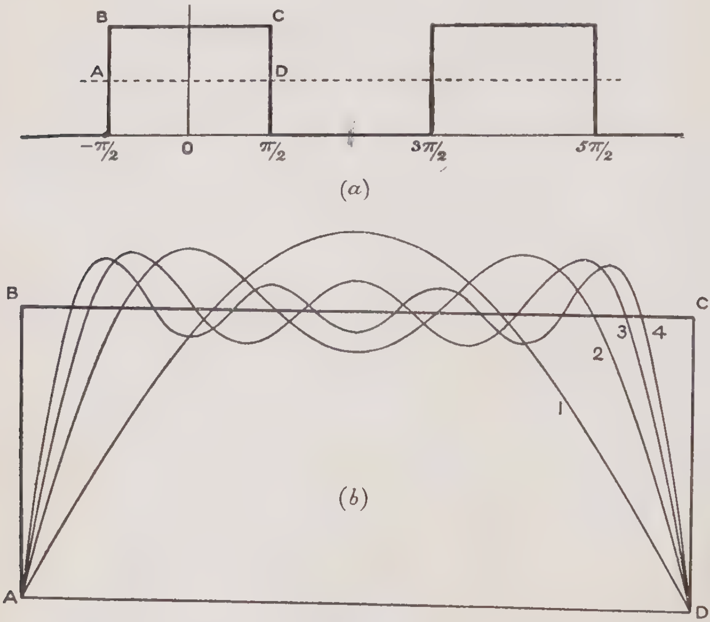


FIG. 9.

Hence substituting the values of the $a$'s in (1) we get

$$f(t) = \frac{2h}{\pi}\left\{\frac{\alpha}{2} + \frac{\sin\alpha}{1}\cos t + \frac{\sin 2\alpha}{2}\cos 2t + \frac{\sin 3\alpha}{3}\cos 3t + \ldots\right\} \quad . \quad . \quad . \quad (2)$$

In particular, if $\alpha = \frac{\pi}{2}$ we get

$$f(t) = \frac{2h}{\pi}\left\{\frac{\pi}{4} + \cos t - \frac{\cos 3t}{3} + \frac{\cos 5t}{5} - \dots\right\}. \quad (3)$$

A sketch of this function is shown in Fig. 9(*a*) where the dotted line indicates the value of the first term. The way in which the successive terms approximate to the portion ABCD is shown enlarged in (*b*) where the curves marked 1, 2, 3 and 4 indicate the sum of the first 2, 3, 4 and 5 terms of (3) respectively.

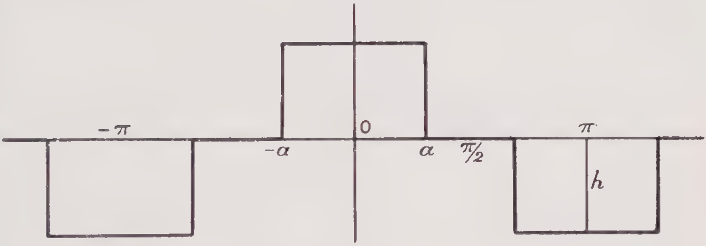


Fig. 10.

We may note here that the constant term $\frac{a_0}{2}$ is always the *mean height of the function;* for the mean height is $\frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt$, which is the value of $\frac{a_0}{2}$.

*Problem II.*—Analyse an even Class IV function which is equal to $h$ from $t = 0$ to $t = \alpha$ and to zero from $t = \alpha$ to $t = \frac{\pi}{2}$.

Here

$$f(t) = a_1 \cos t + a_3 \cos 3t + \dots$$

where

$$a_n = \frac{4}{\pi}\int_0^{\pi/2} f(t) \cos nt\,dt = \frac{4}{\pi}\int_0^{\alpha} h \cos nt\,dt = \frac{4h}{n\pi} \sin n\alpha.$$

Hence

$$f(t) = \frac{4h}{\pi}\left\{\frac{\sin \alpha}{1} \cos t + \frac{\sin 3\alpha}{3} \cos 3t + \frac{\sin 5\alpha}{5} \cos 5t + \ldots\right\} \quad . \quad . \quad . \quad (4)$$

The graph of the function is shown in Fig. 10.

The part from $\frac{\pi}{2}$ to $\pi$ is constructed from the part

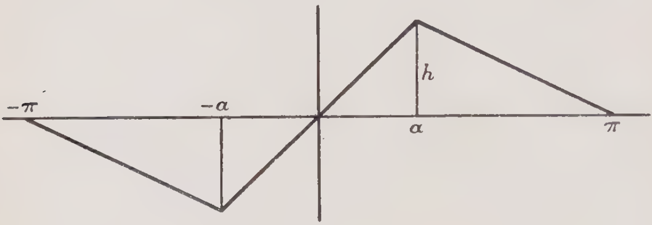


FIG. 11.

from $-\frac{\pi}{2}$ to 0 by means of the relation $f(t + \pi) = -f(t)$, which holds for all Class IV functions.

*Problem III.*—Analyse an odd function represented by $f(t) = \frac{ht}{\alpha}$ from $t = 0$ to $t = \alpha$ and by $h\left(\frac{\pi - t}{\pi - \alpha}\right)$ from $t = \alpha$ to $t = \pi$. This function is represented by Fig. 11.

Unlike the two previous functions, this has no discontinuities in magnitude but only in slope.

Since the function is odd, we have

$$f(t) = b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \ . \ . \ .$$

where
$$b_n = \frac{2}{\pi}\int_0^\pi f(t) \sin nt \, dt.$$

If we integrate this expression by parts we get

$$b_n = \frac{2}{\pi}\left[\frac{f(t)\cos nt}{-n}\right]_0^\pi + \frac{2}{\pi n}\int_0^\pi f'(t)\cos nt\,dt.$$

This step would obviously be justifiable if $f(t)$ was a single function. As it is not, we must break the range of integration up into two, one from 0 to $\alpha$ and the other from $\alpha$ to $\pi$. If we do this we see that the value of the first term at the upper limit of the first range is the same as its value at the lower limit of the second range and hence these cancel, leaving only the values at 0 and $\pi$ as written. We might have said simply, that the first term only takes cognisance of the *magnitude* of $f(t)$; and since $f(t)$ has no abrupt changes in *magnitude* it behaves, as far as the first term is concerned, as an ordinary analytic function. Now $f(t)$ is zero both when $t = 0$ and when $t = \pi$. Hence the first term entirely vanishes. The second term, the integral, breaks up into two; for $f'(t) = \frac{h}{\alpha}$ from 0 to $\alpha$ and $= -\frac{h}{\pi - \alpha}$ from $\alpha$ to $\pi$.

Hence we have

$$b_n = \frac{2}{\pi n}\left\{\frac{h}{\alpha}\int_0^\alpha \cos nt\,dt - \frac{h}{\pi - \alpha}\int_\alpha^\pi \cos nt\,dt\right\}$$

$$= \frac{2h}{\pi n}\left\{\frac{\sin n\alpha}{n\alpha} + \frac{\sin n\alpha}{n(\pi - \alpha)}\right\} = \frac{2h \sin n\alpha}{\alpha(\pi - \alpha)n^2}.$$

Hence

$$f(t) = \frac{2h}{\alpha(\pi - \alpha)}\left\{\frac{\sin \alpha}{1^2}\sin t + \frac{\sin 2\alpha}{2^2}\sin 2t + \frac{\sin 3\alpha}{3^2}\sin 3t + \dots\right\} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Some special cases of this result are worth noticing.

If $\alpha = \frac{\pi}{2}$ we get

$$f(t) = \frac{8h}{\pi^2}\left\{\sin t - \frac{\sin 3t}{3^2} + \frac{\sin 5t}{5^2} - \frac{\sin 7t}{7^2} + \ldots\right\} \quad \ldots (6)$$

so that the function is a Class IV function, which is geometrically obvious from Fig. 11 when $\alpha = \pi/2$.

Again, let us put $\alpha = 0$ so that the function takes the form shown in Fig. 12.

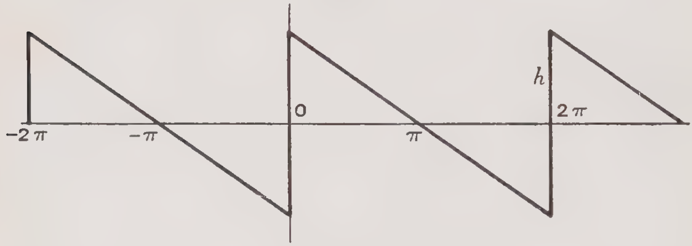


FIG. 12.

The expression (5) requires evaluating as an indeterminate form, since both numerators and denominators are zero. However, on replacing $\frac{\sin n\alpha}{\alpha}$ by $n$ when $\alpha = 0$, we get at once

$$f(t) = \frac{2h}{\pi}\left\{\sin t + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \frac{\sin 4t}{4} + \ldots\right\} \quad \ldots (7)$$

Here the question arises, What is the value of this series when $t = 0$? If we put $t = 0$ we see that the sum of the series is *formally* zero; but it really takes an indeterminate form, as is seen by imagining $t$ infinitesimally small when we get

$$f(t) = \frac{2h}{\pi}\left\{t + t + t + t + t + \ . \ . \ .\right\} = \frac{2ht}{\pi}\infty$$

which is of the form $(0 \times \infty)$.

The value of the *function* at $t = 0$ is also indeterminate: it is obvious from Fig. 12 that when $t = 0$ the function possesses all values from $-h$ to $+h$. We shall prove subsequently (Chapter VI) that the *series* is also indeterminate when $t = 0$ and that the range of indeterminateness of the series is 1·179 times that of the function.

*Problem IV.*—Let us analyse an even function which is represented by the same equations from 0 to $\pi$ as the function in Problem III.

Here $$f(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + \ . \ . \ .$$

where $$a_n = \frac{2}{\pi}\int_0^{\pi} f(t) \cos nt \, dt.$$

If this be integrated by parts as in the last case, we get

$$a_n = \frac{2}{\pi}\left[\frac{f(t) \sin nt}{n}\right]_0^{\pi} - \frac{2}{\pi n}\int_0^{\pi} f'(t) \sin nt \, dt.$$

As before, the first term vanishes at both limits, and the second term gives

$$a_n = -\frac{2h}{\pi n}\left\{\frac{1}{\alpha}\int_0^{\alpha} \sin nt \, dt - \frac{1}{\pi - \alpha}\int_{\alpha}^{\pi} \sin nt \, dt\right\}$$

$$= -\frac{2h}{\pi n^2}\left\{\frac{1 - \cos n\alpha}{\alpha} + \frac{\cos n\pi - \cos n\alpha}{\pi - \alpha}\right\}.$$

This takes different forms according as to whether $n$ is odd or even owing to the $\cos n\pi$ term. If $n$ is even it reduces to $$a_n = -\frac{4h}{\alpha(\pi - \alpha)} \cdot \frac{\sin^2 n\alpha/2}{n^2}$$

and if $n$ is odd, to

$$a_n = \frac{2b}{\alpha(\pi - \alpha)}\left\{\frac{\frac{2\alpha}{\pi} - 1 + \cos n\alpha}{n^2}\right\}.$$

The value of $a_0$ which is twice the mean height of the function is obviously $b$. The shape of the function is represented by Fig. 7($c$), p. 17.

*Problem V.*—From the two previous problems we can at once write down the analysis of the Class III function which is equal to $\frac{bt}{\alpha}$ from $t = 0$ to $\alpha$ and to $\frac{b(\pi - t)}{\pi - \alpha}$ from $t = \alpha$ to $\pi$. For it must be represented by

$$\begin{aligned} f(t) = a_1 \cos t + a_3 \cos 3t + \ldots \\ + b_1 \sin t + b_3 \sin 3t + \ldots \end{aligned}$$

and the values of these coefficients have all been found in the last two Problems, giving the result :

$$\begin{aligned} f(t) = \frac{2b}{\alpha(\pi\alpha)}\Bigg[\frac{\sin \alpha}{1^2} \sin t + \frac{\sin 3\alpha}{3^2} \sin 3t + \ldots \\ + \frac{\frac{2\alpha}{\pi} - 1 + \cos \alpha}{1^2} \cos t + \frac{\frac{2\alpha}{\pi} - 1 + \cos 3\alpha}{3^2} \\ \times \cos 3t + \ldots \Bigg]. \end{aligned}$$

The shape of the function is represented by Fig. 7($d$), p. 17. If in this series we put $\alpha = \frac{\pi}{2}$, all the cosine harmonics vanish and the function becomes, as is geometrically obvious, a Class IV function.

*Problem VI.*—Analyse an odd Class IV function which is equal to $\frac{ht}{a}$ from $t = 0$ to $a$, and to $h$ from $t = a$ to $\frac{\pi}{2}$.

Here

$$f(t) = b_1 \sin t + b_3 \sin 3t + \ldots$$

where

$$b_n = \frac{4}{\pi}\int_0^{\pi/2} f(t) \sin nt \, dt.$$

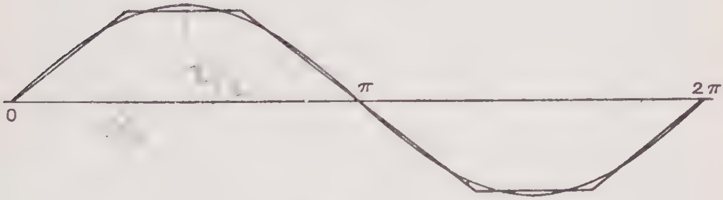

Fig. 13.

Integrating by parts we get

$$b_n = \frac{4}{\pi}\left[\frac{f(t) \cos nt}{-n}\right]_0^{\pi/2} + \frac{4}{\pi n}\int_0^{\pi/2} f'(t) \cos nt \, dt.$$

This is justified, since $f(t)$ has no discontinuities in magnitude. The first term vanishes at both limits; also $f'(t) = \frac{h}{a}$ from 0 to $a$ and is zero afterwards; hence

$$b_n = \frac{4}{\pi n} \cdot \frac{h}{an} \cdot \sin na = \frac{4h \sin na}{\pi a n^2}$$

so that we have

$$f(t) = \frac{4h}{\pi a}\left\{\frac{\sin a}{1} \sin t + \frac{\sin 3a}{3^2} \sin 3t + \frac{\sin 5a}{5^2} \sin 5t + \ldots\right\} \quad . \quad . \quad . \quad (8)$$

If, for example, we take $\alpha = \frac{\pi}{3}$, we get $\sin \alpha = \frac{\sqrt{3}}{2}$; $\sin 3\alpha = 0$; $\sin 5\alpha = -\frac{\sqrt{3}}{2}$, etc., and so

$$f(t) = \frac{6\sqrt{3}}{\pi^2} h\left\{\sin t - \frac{\sin 5t}{5^2} + \frac{\sin 7t}{7^2} - \frac{\sin 11t}{11^2} + \ldots\right\}$$

or $f(t) = h\{1{\cdot}056 \sin t - {\cdot}042 \sin 5t + {\cdot}022 \sin 7t - {\cdot}009 \sin 11t + \ldots\}$.

This function together with its first harmonic is shown in Fig. 13.

*Problem VII.*—Analyse an even Class IV function given by $f(t) = h\left(1 - \frac{4t^2}{\pi^2}\right)$ from $t = 0$ to $t = \frac{\pi}{2}$.

Here $f(t) = a_1 \cos t + a_3 \cos 3t + \ldots$,

where $$a_n = \frac{4}{\pi}\int_0^{\pi/2} f(t) \cos nt\, dt.$$

Since $f(t)$ has no discontinuities in magnitude we have, on integrating by parts,

$$a_n = \frac{4}{\pi}\left[\frac{f(t) \sin nt}{n}\right]_0^{\pi/2} - \frac{4}{\pi n}\int_0^{\pi/2} f'(t) \sin nt\, dt \quad . \quad . \quad (9)$$

The first term vanishes at both limits; $f(t)$ vanishing at $\frac{\pi}{2}$ and $\sin nt$ at zero. Since $f'(t)$ has no discontinuities in magnitude, we may integrate (9) again by parts, giving

$$a_n = -\frac{4}{\pi n}\left[\frac{f'(t) \cos nt}{-n}\right]_0^{\pi/2} - \frac{4}{\pi n^2}\int_0^{\pi/2} f''(t) \cos nt\, dt.$$

Again the first term vanishes at both limits. The integral can easily be evaluated, since $f''(t) = -\frac{8h}{\pi^2}$

from o to $\frac{\pi}{2}$. Hence we get

$$a_n = \frac{32h}{\pi^3 n^2}\int_0^{\pi/2} \cos nt\, dt = \frac{32h}{\pi^3 n^3} \sin \frac{n\pi}{2}.$$

The sine term is $+1$ and $-1$ alternately for $n = 1, 3, 5$, etc. and so

$$f(t) = \frac{32h}{\pi^3}\left\{\cos t - \frac{\cos 3t}{3^3} + \frac{\cos 5t}{5^3} - \frac{\cos 7t}{7^3} + \ldots\right\} \quad (10)$$

This function is shown by the heavy line in Fig. 14.

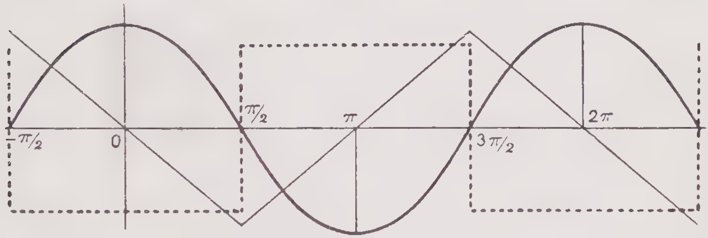


Fig. 14.

**Differentiation of a Fourier Series.** If we differentiate (10) we get

$$f'(t) = \frac{32h}{\pi^3}\left\{- \sin t + \frac{\sin 3t}{3^2} - \frac{\sin 5t}{5^2} + \ldots\right\}.$$

But $f'(t)$ is represented by the thin line in Fig. 14. The maximum numerical value of this function is easily seen to be $\frac{4h}{\pi}$; since $f'(t) = -\frac{8t}{\pi^2}$ from o to $\frac{\pi}{2}$. If we call this maximum value H, the expression for $f'(t)$ becomes

$$f'(t) = -\frac{8H}{\pi^2}\left\{\sin t - \frac{\sin 3t}{3^2} + \frac{\sin 5t}{5^2} - \ldots\right\};$$

which is in agreement with (6), p. 42, for the same function.

If we differentiate this again we get

$$f''(t) = -\frac{8H}{\pi^2}\left\{\cos t - \frac{\cos 3t}{3} + \frac{\cos 5t}{5} \ldots\right\}$$

but $f''(t)$ is the function represented by the dotted lines in Fig. 14, consisting of lines parallel to the axis and at distances $H \div \frac{\pi}{2}$ above and below it. If this distance be called H′, we have

$$f''(t) = -\frac{4H'}{\pi}\left\{\cos t - \frac{\cos 3t}{3} + \ldots\right\}$$

which is in agreement with (3), p. 39, if the axis be shifted parallel to itself so as to make the constant term zero.

These examples illustrate the theorem that we may differentiate a Fourier Series term by term and *provided that the differentiated series is convergent* the result will represent the expansion of the differential coefficient of the given function. This result hardly, we think, requires further proof than is afforded by this example. Conversely, we may in a similar manner, integrate a Fourier series and the result will represent the integral of the given function. No qualification about convergency is necessary here, since the integrated series is necessarily more convergent than the original one.

All the functions we have analysed so far have belonged to one of the four special classes. We now give one example of the general type.

*Problem VIII.*—Analyse the function given by $f(t) = ht/\pi$ from $t = 0$ to $\pi$ and by $f(t) = 0$ from $t = \pi$ to $2\pi$.

Since there is no symmetry about this function, we have

$$f(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + \ldots$$
$$+ b_1 \sin t + b_2 \sin 2t + \ldots$$

where
$$a_n + ib_n = \frac{1}{\pi}\int_0^{2\pi} f(t)e^{int}\,dt$$
$$= \frac{h}{\pi^2}\int_0^{\pi} t\,e^{int}\,dt.$$

Integrating by parts, we get

$$a_n + ib_n = \frac{h}{\pi^2}\left[\frac{t\,e^{int}}{in}\right]_0^{\pi} - \frac{h}{\pi^2 in}\int_0^{\pi} e^{int}\,dt$$
$$= \frac{h}{\pi^2}\cdot\frac{\pi e^{in\pi}}{in} + \frac{h}{\pi^2 n^2}\left(e^{in\pi} - e^{in0}\right).$$

Remembering that $e^0 = 1$ and $e^{in\pi} = \cos n\pi = (-1)^n$, we get, on equating real and imaginary parts,

$$a_n = \frac{h}{\pi^2 n^2}\left\{(-1)^n - 1\right\} \text{ and } b_n = \frac{h}{\pi n}(-1)^{n-1}.$$

The constant term $\frac{a_0}{2}$ is obviously $\frac{h}{4}$, so we have

$$f(t) = h\left\{\frac{1}{4} - \frac{2}{\pi^2}\left(\frac{\cos t}{1} + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} \ldots\right)\right.$$
$$\left. + \frac{1}{\pi}\left(\frac{\sin t}{1} - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \ldots\right)\right\}.$$

Hitherto our artificial functions have been composed of straight lines or parabolas; we will now take some cases where they are composed of segments of sine curves. In the next two problems it will be convenient to take the period a sub-multiple of $2\pi$.

*Problem IX.*—Analyse the rectified sine curve. The function consists of a repetition of the positive halves of the sine curve $y = h \sin t$. See Fig. 15.

Here the period is $\pi$ and the function is an even function, so that

$$f(t) = \frac{a_0}{2} + a_2 \cos 2t + a_4 \cos 4t + \ldots$$

where $$a_n = \frac{2}{\pi}\int_0^{\pi} h \sin t \cos nt \, dt.$$

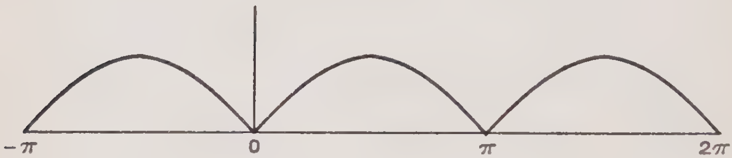

FIG. 15.

But the curve $h \sin t$ is symmetrical about an axis of $y$ through $t = \frac{\pi}{2}$, and so also is $\cos nt$ when $n$ is even; hence we may take the integral from 0 to $\frac{\pi}{2}$, giving

$$a_n = \frac{4h}{\pi}\int_0^{\pi/2} \sin t \cos nt \, dt$$

$$= \frac{2h}{\pi}\int_0^{\pi/2}\left\{\sin (n+1)t - \sin(n-1)t\right\}dt$$

$$= \frac{2h}{\pi}\left[\frac{\cos (n-1)t}{n-1} - \frac{\cos (n+1)t}{n+1}\right]_0^{\pi/2}$$

Since $n$ is even, $n - 1$ and $n + 1$ are both odd, and

therefore both terms vanish at the upper limit so that we have

$$a_n = \frac{2h}{\pi}\left\{-\frac{1}{n-1}+\frac{1}{n+1}\right\} = \frac{-4h}{\pi(n^2-1)}.$$

This assumes no indeterminate form when $n = 0$; hence

$$f(t) = \frac{4h}{\pi}\left\{\frac{1}{2} - \frac{\cos 2t}{1\cdot 3} - \frac{\cos 4t}{3\cdot 5} - \frac{\cos 6t}{5\cdot 7} - \ldots\ldots\right\}. \qquad \ldots \quad (11)$$

The student of electrical engineering will recognise the rectified sine curve as the voltage wave form of a

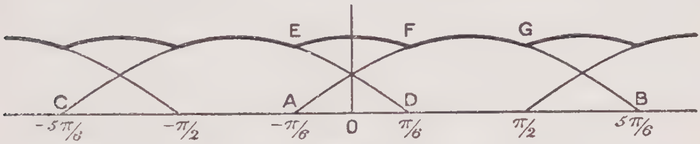


FIG. 16.

simple dynamo with a single coil in the armature and a two-piece commutator. The next problem will give the voltage wave form of an armature with three coils at 120° apart and a three sector commutator.*

*Problem X.*—The positive halves of the sine wave $y = h \sin t$ are drawn at intervals of $2\pi/3$ and the ordinates added where they overlap. Analyse the resulting curve.

This curve is shown in Fig. 16.

In order to add the overlapping curves, we will draw the axis of $y$ through a point of intersection as shown. The equation of the arc AB is then $y = h \sin\left(t + \frac{\pi}{6}\right)$ and of CD is $y = h \sin\left(\frac{\pi}{6} - t\right)$. The

* The "field form" is supposed to be sinusoidal in both these problems.

sum of these two is $y = 2h \sin \frac{\pi}{6} \cos t = h \cos t$, and this holds from $-\frac{\pi}{6}$ to $+\frac{\pi}{6}$ and is the equation of the arc EF. But the equation of the arc FG would be $y = h \cos t$ if the axis of $y$ passed through its mid point; so that both arcs are similar and hence the required periodic function has a period of $\pi/3$ or $2\pi/6$. The function is also an even function, so that, using the formulæ for the period an aliquot part of $2\pi$ on p. 29, we have

$$f(t) = \frac{a_0}{2} + a_1 \cos 6t + a_2 \cos 12t + \dots$$

where
$$a_n = \frac{12}{\pi} \int_0^{\pi/6} f(t) \cos nt \, dt$$

$$= \frac{12h}{\pi} \int_0^{\pi/6} \cos t \cos 6nt \, dt$$

$$= \frac{6h}{\pi} \int_0^{\pi/6} \{\cos (6n - 1)t + \cos (6n + 1)t\} dt$$

$$= \frac{6h}{\pi} \left[ \frac{\sin (6n - 1)t}{6n - 1} + \frac{\sin (6n + 1)t}{6n + 1} \right]_0^{\pi/6}$$

$$= \frac{6h}{\pi} \left\{ \frac{\sin (n\pi - \pi/6)}{6n - 1} + \frac{\sin (n\pi + \pi/6)}{6n + 1} \right\}.$$

Both sine terms are numerically equal to $\sin \pi/6$ or $1/2$, and both are of opposite sign, so they give $a_n = \frac{6h}{\pi(36n^2 - 1)}$ numerically, but with terms alternately + and − starting with $a_1$ +; $a_0$ is also obtained by putting $n = 0$ in this expression.

Hence we have

$$f(t) = \frac{6h}{\pi}\left\{\frac{1}{2} + \frac{\cos 6t}{5 \cdot 7} - \frac{\cos 12t}{11 \cdot 13} + \frac{\cos 18t}{17 \cdot 19} - \ldots\right\}$$

$$= h\{0{\cdot}955 + 0{\cdot}055 \cos 6t - 0{\cdot}013 \cos 12t$$
$$+ 0{\cdot}006 \cos 18t - \ldots\}$$

which shows that the amplitude of the lowest harmonic is less than 6% of the amplitude of the constant term.

We will now take two mechanical problems leading to artificial functions composed of arcs of sine curves.

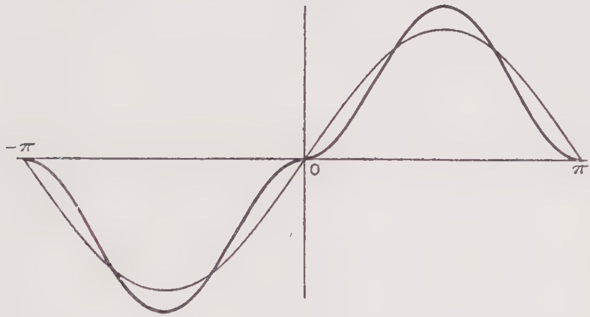


Fig. 17.

*Problem XI.*—A body performs simple harmonic motion in a fluid in which the resistance is proportional to the square of the velocity. Analyse the periodic function representing the resistance to the motion.

If the velocity is represented by $k \sin t$, the numerical value of the resistance will be given by $h \sin^2 t$; but since the resistance always opposes the motion, it will change sign whenever the velocity reverses, so the resistance will be a Class IV function represented graphically by Fig. 17 where also the thin line shows the first harmonic.

Here $f(t)$ is represented by $h \sin^2 t$ from $t = 0$ to $\pi$; hence

$$f(t) = b_1 \sin t + b_3 \sin 3t + \ldots$$

where $$b_n = \frac{4h}{\pi}\int_0^{\pi/2} \sin^2 t \sin nt\, dt$$

$$= \frac{2h}{\pi}\int_0^{\pi/2} (1 - \cos 2t) \sin nt\, dt.$$

The first term of this integral gives

$$\frac{2h}{\pi}\left[\frac{\cos nt}{-n}\right]_0^{\pi/2};$$

which, since $n$ is necessarily odd, is equal to $2h/\pi n$. The second term gives

$$-\frac{h}{\pi}\int_0^{\pi/2}\{\sin (n+2)t + \sin (n-2)t\}dt$$

$$= \frac{h}{\pi}\left[\frac{\cos (n+2)t}{n+2} + \frac{\cos (n-2)t}{n-2}\right]_0^{\pi/2}.$$

Remembering that $n$ is odd, this gives $-\frac{h}{\pi}\left(\frac{1}{n+2} + \frac{1}{n-2}\right) = \frac{-2nh}{\pi(n^2-4)}$. If this be added to the first term we get $b_n = \frac{-8h}{\pi n(n^2-4)}$ and so

$$f(t) = \frac{8h}{\pi}\left\{\frac{\sin t}{3} - \frac{\sin 3t}{1.3.5} - \frac{\sin 5t}{3.5.7} - \frac{\sin 7t}{5.7.9} - \dots\right\}.$$

The loss of symmetry in this expression due to all the terms being negative save the first is in appearance only: really the factors in the first denominator are $-1$, $1$ and $3$.

*Problem XII.*—A reservoir is divided into two by a vertical partition with a *small* hole in it near the bottom. The height of liquid in one part is kept constant and may be taken as zero; the height in the

other part varies according to $k \cos t$: analyse the function representing the velocity of flow through the hole.

The velocity of flow is proportional to the square root of the numerical difference of heights and changes sign whenever this changes. So we have to analyse an even Class IV function of period $2\pi$ which is equal to $h\sqrt{\cos t}$ from $t = 0$ to $t = \frac{\pi}{2}$. This function is shown by the heavy line in Fig. 18,* the thin line representing the first harmonic.

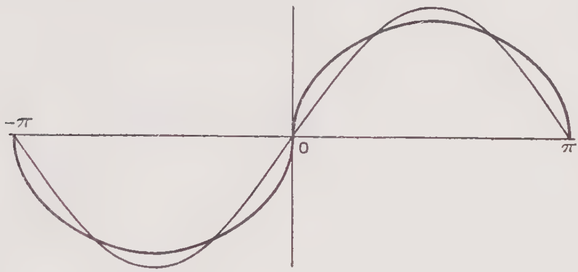


FIG. 18.

Here $\quad f(t) = a_1 \cos t + a_3 \cos 3t + \ldots$

where $$a_n = \frac{4h}{\pi}\int_0^{\pi/2} \cos^{\frac{1}{2}} t \cos nt\, dt.$$

A Reduction Formula may be found for this integral which will make $a_n$ depend upon $a_{n-2}$, and thus finally upon $a_1$. Consider $\frac{d}{dt}(\cos^{3/2} t \sin nt)$, where $n$ is even. It is equal to

$$-\frac{3}{2}\cos^{\frac{1}{2}} t \sin t \sin nt + n \cos^{3/2} t \cos nt$$

* By an oversight it is represented as an odd function in the diagram.

$$= \cos \tfrac{1}{2}t\left\{n \cos t \cos nt - \frac{3}{2} \sin t \sin nt\right\}$$

$$= \frac{\cos \frac{1}{2}t}{4}\left\{(2n + 3) \cos (n + 1)t + (2n - 3) \cos (n - 1)t\right\}.$$

Now integrate both sides of this from $t = 0$ to $t = \frac{\pi}{2}$.

The left-hand side becomes $\Big[\cos^{3/2} t \sin nt\Big]_0^{\pi/2}$ which is zero since it vanishes at both limits. Hence we have

$$\int_0^{\pi/2} \cos^{\frac{1}{2}} t \cos (n+1)t\, td = -\frac{2n-3}{2n+3}\int_0^{\pi/2} \cos^{\frac{1}{2}} t \cos (n-1)t\, dt.$$

Putting $n = 2$ we have $a_3 = -\frac{1}{7}a_1$; putting $n = 4$ we obtain $a_5 = -\frac{5}{11}a_3 = +\frac{1}{7}\cdot\frac{5}{11}a_1$, and so on.

Hence
$$f(t) = a_1\left\{\cos t - \frac{1}{7}\cos 3t + \frac{1}{7}\cdot\frac{5}{11}\cos 5t - \frac{1}{7}\cdot\frac{5}{11}\cdot\frac{9}{15}\cos 7t + \dots\right\}$$

where
$$a_1 = \frac{4h}{\pi}\int_0^{\pi/2} \cos^{3/2} t\, dt.$$

This integral can easily be evaluated by Simpson's rule, taking about 5 intermediate ordinates. It will be found that $a_1 = 1{\cdot}113h$.*

* The student familiar with the Gamma Function will know that $\int_0^{\pi/2} \cos^{3/2}\theta d\theta = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(5/4)}{\Gamma(7/4)}$. (See Williamson's *Integral Calculus*, Chapter VI and the Table of $\log \Gamma(p)$ at the end of the chapter.) We may point out that when $n$ is large $a_n$ approximates to $\pm h\sqrt{\frac{2}{\pi n^3}}$.

If we integrate the expression for $f(t)$ we shall obtain a result proportional to the total flow through the hole up to the time $t$, viz.

$$1{\cdot}113b\left\{\sin t - \frac{\sin 3t}{21} + \frac{\sin 5t}{77} - \frac{3 \sin 7t}{539} + \ldots\right\}.$$

**A General Formula for the Fourier Constants in terms of the Discontinuities.** We shall now obtain a very general formula for the Fourier constants from which the expansions in nearly all the previous examples can be written down straight away. Let $f(t)$ be a periodic function of period $2\pi$ represented in different portions of its range by straight lines or by arcs of parabolas of any finite degree.*

Let the function, or any of its differential coefficients, have discontinuities of magnitudes $I_\alpha$, $I_\beta$, $I_\gamma$ . . . $I'_\alpha$, $I'_\beta$ . . . $I''_\alpha$, $I''_\beta$ . . . at the points $\alpha$, $\beta$, $\gamma$ . . . using the notation of Chapter I, p. 14. We will suppose, to save extra words in the course of the proof, that there are no discontinuities at $t = 0$; although, since the origin is arbitrary, it is obvious that such merely require treating like any of the other discontinuities.

We have $f(t) = \dfrac{a_0}{2} + a_1 \cos t + \ldots + b_1 \sin t + \ldots$

where

$$a_n + ib_n = \frac{1}{\pi}\int_0^{2\pi} f(t)e^{int}\,dt.$$

Integrating by parts we get

$$\frac{1}{\pi}\left[\frac{f(t)e^{int}}{in}\right] - \frac{1}{\pi in}\int f'(t)e^{int}\,dt \quad . \quad . \quad (12)$$

* A parabola of the $p$th degree is the curve given by the equation $y = c_0 + c_1 t + c_2 t^2 + \ldots + c_p t^p$.

Let now $f(t)$ be represented by one analytic function from 0 to $\alpha$ and by another from $\alpha$ to $2\pi$. The first term when taken between limits gives

$$\frac{1}{\pi in}\left\{f(\alpha - 0)e^{in\alpha} - f(0)e^{in0} + f(2\pi)e^{in2\pi} - f(\alpha + 0)e^{in\alpha}\right\}$$

By supposition there is no discontinuity at 0 and so $f(2\pi) = f(0)$; also $e^{2\pi in} = e^{in0}$, so we are left with $\frac{-e^{in\alpha}}{\pi in}\left\{f(\alpha + 0) - f(\alpha - 0)\right\}$;

that is, with $$+\frac{i\mathrm{I}_\alpha e^{in\alpha}}{\pi n}.$$

If there are also other discontinuities at $\beta$, $\gamma$, etc., it is clear, by breaking up the range of integration into ranges from 0 to $\alpha$, $\alpha$ to $\beta$, etc., that the first term of (12) will become $\frac{i}{\pi n}\Sigma\mathrm{I}_\alpha e^{in\alpha}$, where the summation covers all the discontinuities of $f(t)$ in *magnitude*.

If we similarly integrate the second term of (12) by parts we get

$$\frac{-1}{\pi in}\left[\frac{f'(t)e^{int}}{in}\right] + \frac{1}{\pi i^2n^2}\int f''(t)e^{int}\,dt \quad . \quad . \quad (13)$$

In an exactly similar manner the value of the first term of this expression is

$$-\frac{1}{\pi n^2}\Sigma\mathrm{I}'_\alpha e^{in\alpha}$$

where $\mathrm{I}'_\alpha$ is the discontinuity of $f'(t)$ at $t = \alpha$ and the summation extends to all such discontinuities. Similarly, if we integrate the second term of (13) by parts we get

$$-\frac{1}{\pi n^2}\left[\frac{f''(t)e^{int}}{in}\right] + \frac{1}{\pi in^3}\int f'''(t)e^{int}\,dt.$$

The value of the first term of this is similarly

$$-\frac{i}{\pi n^3}\Sigma \mathrm{I}''_\alpha e^{in\alpha}.$$

Proceeding in this manner, we get the final result that

$$\pi(a_n + ib_n) = \frac{i}{n}\Sigma \mathrm{I}_\alpha e^{in\alpha} - \frac{1}{n^2}\Sigma \mathrm{I}'_\alpha e^{in\alpha} - \frac{i}{n^3}\Sigma \mathrm{I}''_\alpha e^{in\alpha} + \frac{1}{n^4}\Sigma \mathrm{I}'''_\alpha e^{in\alpha} + \quad . \quad . \quad (14)$$

This series terminates; for if $p$ is the highest degree of any of the parabolic arcs, $f^{(p+1)}(t)$ is everywhere zero, so there are no discontinuities, $\mathrm{I}_\alpha^{p+1}$ or any higher ones.

Separating real and imaginary parts we get

$$\pi a_n = -\frac{1}{n}\Sigma \mathrm{I}_\alpha \sin n\alpha - \frac{1}{n^2}\Sigma \mathrm{I}'_\alpha \cos n\alpha + \frac{1}{n^3}\Sigma \mathrm{I}''_\alpha \sin n\alpha + \frac{1}{n^4}\Sigma \mathrm{I}'''_\alpha \cos n\alpha - \quad . \quad . \quad . \quad (15)$$

and

$$\pi b_n = \frac{1}{n}\Sigma \mathrm{I}_\alpha \cos n\alpha - \frac{1}{n^2}\Sigma \mathrm{I}'_\alpha \sin n\alpha - \frac{1}{n^3}\Sigma \mathrm{I}''_\alpha \cos n\alpha + \frac{1}{n^4}\Sigma \mathrm{I}'''_\alpha \sin n\alpha - \quad . \quad . \quad . \quad (16)$$

As an example, we will apply this last equation to the function in Problem I. Here we have a discontinuity $+h$ at $-\alpha$ and one of $-h$ at $+\alpha$; and there are no discontinuities in any of the differential coefficients, these all being zero throughout. Hence

$$\pi a_n = -\frac{1}{n}h \sin(-n\alpha) - \frac{1}{n}(-h) \sin n\alpha = \frac{2h}{n} \sin n\alpha,$$

the same as previously found.

Again, let us take the function in Problem III.

Here only $f'(t)$ has any discontinuities; and has a discontinuity of magnitude $\frac{h}{a} + \frac{h}{\pi - a}$ or $\frac{h\pi}{a(\pi - a)}$ at $-a$ and one of magnitude $\frac{-h\pi}{a(\pi - a)}$ at $+a$.

Hence by (16),

$$\pi b_n = -\frac{1}{n^2}\frac{h\pi}{a(\pi - a)}\sin(-na) + \frac{1}{n^2}\frac{h\pi}{a(\pi - a)}\sin na$$

or
$$b_n = \frac{2h}{a(\pi - a)} \cdot \frac{\sin na}{n^2}$$

again in agreement with the value previously found. Formulæ (15) and (16) can be simplified in the case of functions belonging to the four special classes by consideration of the relation of the discontinuity at $-a$ to that at $+a$: we leave this to the student.

The formulæ (15) and (16) may be employed even when they do not terminate, provided that they are convergent. As an example, we will consider the rectified sine curve dealt with in Problem IX, save that we will now make the period $2\pi$ instead of $\pi$. The function is then represented by $h \sin t/2$ from $t = 0$ to $t = 2\pi$.

The function itself has no discontinuities in magnitude, but its differential coefficient has one of magnitude $\frac{2h}{2}$ at $t = 0$; so that $I'_0 = h$. Its second order differential coefficient has no discontinuities in magnitude, its third has one of magnitude $-\frac{2h}{8}$ at $t = 0$; so that $I'''_0 = -\frac{h}{4}$; similarly, $I^{\mathrm{V}}_0 = \frac{h}{16}$; $I^{\mathrm{VII}}_0 = -\frac{h}{64}$, etc., whence we get by (15)

$$\begin{aligned}\pi a_n &= -\frac{1}{n^2}h + \frac{1}{n^4}\left(-\frac{h}{4}\right) - \frac{1}{n^6}\left(\frac{h}{16}\right)\ \dots \\ &= -\frac{h}{n^2}\left(1 + \frac{1}{4n^2} + \frac{1}{16n^4} + \frac{1}{64n^6} + \dots\right) \\ &= -\frac{h}{n^2\left(1 - \dfrac{1}{4n^2}\right)} = -\frac{4h}{4n^2 - 1}.\end{aligned}$$

Since the constant term is $\frac{2h}{\pi}$, we get

$$f(t) = \frac{4h}{\pi}\left\{\frac{1}{2} - \frac{\cos t}{1 \cdot 3} - \frac{\cos 2t}{3 \cdot 5} - \frac{\cos 3t}{5 \cdot 7}\ \dots\right\}$$

also in agreement with the previous result.

## EXAMPLES.

1. Prove that the odd function of period $2\pi$ which is equal to $h$ from $t = 0$ to $t = \pi$ is

$$f(t) = \frac{4h}{\pi}\left\{\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \frac{\sin 7t}{7} + \dots\right\}.$$

2. Prove that the even function of period $2\pi$ which is equal to $t^2$ from $t = 0$ to $\pi$ is

$$f(t) = 4\left\{\frac{\pi^2}{12} - \cos t + \frac{\cos 2t}{2^2} - \frac{\cos 3t}{3^2} + \frac{\cos 4t}{4^2} - \dots\right\}.$$

3. (*a*) By putting $t = \pi$ in Example 2 obtain the relation $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$ .

(*b*) By putting $t = \frac{\pi}{2}$ in (6), p. 42, obtain the result $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$ .

(*c*) By putting $t = 0$ in (10), p. 47, show that $\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots$ .

(*d*) By putting $t = \frac{\pi}{2}$ in the result of Example 4 (below) show that $\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots$ .

4. Prove that the odd Class IV function represented by $f(t) = b\left(\frac{3t}{\pi} - \frac{4t^3}{\pi^3}\right)$ from $t = 0$ to $\frac{\pi}{2}$ is

$$f(t) = \frac{96b}{\pi^4}\left\{\sin t - \frac{\sin 3t}{3^4} + \frac{\sin 5t}{5^4} - \frac{\sin 7t}{7^4} + \dots\right\}.$$

5. Four circles are drawn on the sides of a square as diameter. Prove that the polar equation of the quatrefoil defined by the four external semicircles is

$$r = \frac{4a\sqrt{2}}{\pi}\left\{\frac{1}{2} + \frac{\cos 4\theta}{3 \cdot 5} - \frac{\cos 8\theta}{7 \cdot 9} + \frac{\cos 12\theta}{11 \cdot 13} - \dots\right\}$$

where $a$ is the side of the square.

6. Prove that the periodic functions of period $2\pi$ represented by $\sin mt$ and $\cos mt$ from $-\pi$ to $\pi$ are

$$\sin mt = \frac{2}{\pi}\sin m\pi\left\{\frac{\sin t}{1^2 - m^2} - \frac{2\sin 2t}{2^2 - m^2} + \frac{3\sin 3t}{3^2 - m^2} - \dots\right\}$$

and

$$\cos mt = \frac{2}{\pi}\sin m\pi\left\{\frac{1}{2m} + \frac{m\cos t}{1^2 - m^2} - \frac{m\cos 2t}{2^2 - m^2} + \dots\right\}$$

respectively.

7. A long uniform chain hangs over a number of small smooth pegs at a distance $2\pi$ apart in a horizontal line. Each segment hangs in a catenary whose equation is $y = \lambda \cosh x/\lambda$.

Prove that the complete form of the chain is given by

$$y = \frac{2\lambda^2}{\pi}\sinh\frac{\pi}{\lambda}\left\{\frac{1}{2} - \frac{\cos x}{1 + \lambda^2} + \frac{\cos 2x}{1 + 4\lambda^2} - \frac{\cos 3x}{1 + 9\lambda^2} + \dots\right\}.$$

Obtain this result, ($a$) by evaluating the integrals in the usual way with the help of the integral $\int e^{cx}\cos nx\,dx$; and ($b$) by (15), p. 59, from the discontinuities of the function at $\pi$.

8. Prove, by considering its discontinuities, that the odd Class IV function of period $2\pi$ which is represented from 0 to $\alpha$ by the line joining the points (0, 0) and ($\alpha$, $h$); is represented from $\alpha$ to $\beta$ by the line joining the points ($\alpha$, $h$) and ($\beta$, $k$), and from $\beta$ to $\frac{\pi}{2}$ by a line parallel to the axis, is given by

$$f(t) = b_1 \sin t + b_3 \sin 3t + b_5 \sin 5t + \dots$$

where

$$b_n = \frac{4}{\pi n^2}\left\{\left(\frac{h}{\alpha} - \frac{k - h}{\beta - \alpha}\right)\sin n\alpha + \left(\frac{k - h}{\beta - \alpha}\right)\sin n\beta\right\}.$$

9. Obtain the values of the coefficients in Ex. 8 when $\alpha = \frac{\pi}{6}$, $\beta = \frac{\pi}{3}$, $h = \frac{1}{2}$ and $k = \frac{\sqrt{3}}{2}$ so that the points lie on a sine wave.

10. Prove that the even Class IV function of period $2\pi$ which is equal to $\cos^p t$ from $t = 0$ to $t = \pi/2$ is given by

$$(t) = C\left\{\cos t - \frac{1-p}{3+p}\cos 3t + \frac{1-p}{3+p}\cdot\frac{3-p}{5+p}\cos 5t \right.$$
$$\left. - \frac{1-p}{3+p}\cdot\frac{3-p}{5+p}\cdot\frac{5-p}{7+p}\cos 7t + \dots\right\}$$

where $$C = \frac{4}{\pi}\int_0^{\pi/2} \cos^{p+1} t \, dt.\text{*}$$

[Consider $\frac{d}{dt}(\cos^{p+1} t \sin nt)$ in a similar manner to that adopted in Problem XII.]

Note that when $p$ is odd this series terminates; also that the given function is then represented by $\cos^p t$ *throughout its whole range* so that we have expressed $\cos^p t$ when $p$ is odd in a Fourier Series.

11. A periodic function of period $2\pi$ has no discontinuities of any kind except at $t = \pi$; that is, it is represented by one and the same analytic function from $-\pi$ to $+\pi$. Prove that its Fourier constants are given by

$$\pi a_n = (-1)^{n-1}\frac{I'_\pi}{n^2} + (-1)^{n-2}\frac{I'''_\pi}{n^4} + (-1)^{n-3}\frac{I^{v}_\pi}{n^6} + \dots$$

and

$$\pi b_n = (-1)^n \frac{I_\pi}{n} + (-1)^{n-1}\frac{I''_\pi}{n^3} + (-1)^{n-2}\frac{I^{iv}_\pi}{n^5} + \dots$$

where $$I_\pi^{(n)} = f^{(n)}(-\pi) - f^{(n)}(\pi).$$

Obtain the expansion when $f(t) = ce^{t/\lambda}$.

---

* The value of this integral in terms of Gamma Functions is $\frac{2}{\sqrt{\pi}}\,\frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{p}{2}+\frac{3}{2}\right)}$. (Williamson's *Integral Calculus*, Chap. VI.)

12. Prove that the odd Class IV function of period $2\pi$ represented by $h \sin ct$ from $t = 0$ to $t = \frac{\pi}{2}$ is

$$f(t) = \frac{4ch}{\pi} \cos \frac{c\pi}{2} \left\{ \frac{\sin t}{1 - c^2} - \frac{\sin 3t}{9 - c^2} + \frac{\sin 5t}{25 - c^2} - \cdots \right\}.$$

By making $c$ tend to zero while $hc$ remains finite, obtain the expansion when $f(t) = t$ from $t = 0$ to $\frac{\pi}{2}$.

13. Prove that the harmonic analysis of the even function which is equal to $h \cos t$ from $t = 0$ to $t = \frac{\pi}{2}$ and to zero from $t = \frac{\pi}{2}$ to $t = \pi$ is

$$f(t) = \frac{2h}{\pi} \left\{ \frac{1}{2} + \frac{\pi}{4} \cos t + \frac{\cos 2t}{1 \cdot 3} - \frac{\cos 4t}{3 \cdot 5} + \frac{\cos 6t}{5 \cdot 7} \cdots \right\};$$

all odd harmonics being absent except the first.

14. The positive halves of the curve $h \cos t$ are repeated at intervals of $\frac{2\pi}{n}$. Prove, by the result of the previous example, that the harmonic analysis of the sum of these curves is

$$\frac{2nh}{\pi} \left\{ \frac{1}{2} + \frac{\cos 2nt}{(2n - 1)(2n + 1)} - \frac{\cos 4nt}{(4n - 1)(4n + 1)} + \frac{\cos 6nt}{(6n - 1)(6n + 1)} - \cdots \right\}$$

if $n$ is odd and is

$$\frac{2nh}{\pi} \left[ \frac{1}{2} + (-1)^{n/2 - 1} \left\{ \frac{\cos nt}{(n - 1)(n + 1)} - \frac{\cos 2nt}{(2n - 1)(2n + 1)} + \frac{\cos 3nt}{(3n - 1)(3n + 1)} - \cdots \right\} \right]$$

if $n$ is even.

Hence note that the voltage from a commutator with $2n + 1$ sectors is exactly the same as that from one with twice the number of sectors.

15. Two sets of equidistant ordinates at distances of $\frac{\pi}{m}$ apart are drawn, the first set commencing at $t = 0$ and the second set at $t = \frac{\pi}{2m}$, so that the members of the latter are midway between those of the former. Through all the points of intersection of the first set with the curve $f(t) = h \sin t$ short lines are drawn, parallel to the axis of $t$, terminating at the adjacent ordinates of the second set.

Prove that the harmonic analysis of the step-like sine function defined in this manner is

$$f(t) = \frac{2mh}{\pi} \sin \frac{\pi}{2m} \left\{ \sin t - \frac{\sin (2m - 1)t}{2m - 1} - \frac{\sin (2m + 1)t}{2m + 1} \right.$$
$$\left. + \frac{\sin (4m - 1)t}{4m - 1} + \frac{\sin (4m + 1)t}{4m + 1} - \dots \right\}.$$

16. A body performs oscillations in a medium in which the resistance is proportional to the square of the velocity so that the equation of motion is

$$\frac{d^2y}{dt^2} \pm k\left(\frac{dy}{dt}\right)^2 + n^2 y = 0,$$

the sign changing every half oscillation.

Prove, from the result of Problem XI, that, if the damping is so small that its effect on the period may be neglected, the motion is given by

$$y = \frac{A_0 \sin (nt + \gamma)}{1 + \dfrac{4A_0 knt}{3\pi}};$$

where $A_0$ and $\gamma$ are arbitrary constants. How do you explain the fact that when $t$ is large this result is approximately independent of the initial amplitude $A_0$?

17. Show that the result of Ex. 16 may be written

$$y = \frac{3\pi}{4kn(t + t_0)} \sin (nt + \gamma);$$

where $-t_0$ is the time at which the amplitude would have been infinite if the result was valid for large amplitudes.

# CHAPTER IV

## ON THE REPRESENTATION OF PERIODIC ANALYTIC FUNCTIONS BY FOURIER'S SERIES

It is clear that any algebraic function of sines or cosines of $\theta$ and/or integral multiples of $\theta$ must, since each of the sines and cosines is periodic in the interval $2\pi$, likewise be periodic in that interval, and therefore must be expressible in a Fourier's series. The Fourier constants will, of course, be given by the integrals in the usual manner; but they may be obtained in a much simpler way in these cases, for the Fourier expansion is nothing more than another representation of the same function which is already defined for all values of $\theta$ by the single given expression. It must be possible, therefore, to transform the given expression into the Fourier series by mere algebraical manipulation. Thus instead of obtaining the expansion by evaluating the integrals we may use the expansion to obtain the values of the integrals.

To effect this algebraical transformation, we replace the sines and cosines by their exponential values; then, for brevity, denote $e^{i\theta}$ by a single letter, say $u$, and then arrange the expression in positive and negative powers of $u$, grouping $u^n$ with $u^{-n}$. We finally replace $u^n + u^{-n}$ by $2 \cos n\theta$ and $u^n - u^{-n}$ by $2i \sin n\theta$. A few examples will make the method clear.

*Problem I.*—Express, in the form of a Fourier's series, the inverse square of the distance between the two points given by $(r,\theta)$ and $(a,o)$ in Polar Co-ordinates.

The distance squared is $a^2 - 2ar\cos\theta + r^2$, so that we require to represent

$$\frac{1}{a^2 - 2ar\cos\theta + r^2} = \frac{1}{a^2 - ar(u + u^{-1}) + r^2}$$

$$= \frac{u}{(a - ur)(au - r)}$$

in ascending and descending powers of $u$. By the method of Partial Fractions, the expression can be written in the form

$$\frac{P}{a - ur} + \frac{Q}{au - r},$$

where P and Q are independent of $u$.

Adding these fractions and equating to the given expression we get

$$Qa - Pr = 0$$

and $$Pa - Qr = 1.$$

Solving these for P and Q we get for the required expression

$$\frac{1}{a^2 - r^2}\left(\frac{a}{a - ur} + \frac{r}{au - r}\right).$$

If $r$ is less than $a$ this may be written

$$\frac{1}{a^2 - r^2}\left(\frac{1}{1 - ur/a} + \frac{r}{au}\cdot\frac{1}{1 - r/au}\right),$$

the expansion of which is

$$\frac{1}{a^2 - r^2}\left\{1 + \left(\frac{r}{a}\right)u + \left(\frac{r}{a}\right)^2 u^2 + \left(\frac{r}{a}\right)^3 u^3 + \dots \right.$$
$$\left. + \left(\frac{r}{a}\right)u^{-1} + \left(\frac{r}{a}\right)^2 u^{-2} + \dots \right\}$$

$$= \frac{1}{a^2 - r^2}\cdot\left\{1 + 2\left(\frac{r}{a}\right)\cos\theta + 2\left(\frac{r}{a}\right)^2\cos 2\theta \right.$$
$$\left. + 2\left(\frac{r}{a}\right)^3\cos 3\theta + \dots \right\},$$

which is the Fourier series required. It is convergent, since $r/a$ is less than unity. If $r/a > 1$, we can obtain in a similar manner the expansion

$$\frac{1}{r^2 - a^2} \cdot \left\{1 + 2\left(\frac{a}{r}\right) \cos \theta + 2\left(\frac{a}{r}\right)^2 \cos 2\theta + \ldots \right\}.$$

*Problem II.*—To obtain the Fourier series representing $\cos^n \theta$ and $\sin^n \theta$.

These expansions take a different form according to whether $n$ is even or odd.

We have

$$\cos^{2n} \theta = \left(\frac{u + u^{-1}}{2}\right)^{2n}$$

$$= \frac{1}{2^{2n}}\left\{u^{2n} + 2nu^{2n-2} + \frac{2n(2n-1)}{\underline{|2}}u^{2n-4} + \ldots \right.$$

$$+ \frac{\underline{|2n}}{\underline{|n-1}\,\underline{|n+1}}u^2 + \frac{\underline{|2n}}{\underline{|n}\,\underline{|n}}u^\circ + \frac{\underline{|2n}}{\underline{|n+1}\,\underline{|n-1}}u^{-2}$$

$$\left. + \ldots + u^{-2n}\right\}$$

$$= \frac{\underline{|2n}}{2^{2n-1}\,\underline{|n}\,\underline{|n}}\left\{\frac{1}{2} + \frac{n}{n+1} \cos 2\theta + \frac{n(n-1)}{(n+1)(n+2)} \cos 4\theta \right.$$

$$\left. + \frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)} \cos 6\theta + \ldots \right\}.$$

In a similar manner we have

$$\cos^{2n+1} \theta = \frac{\underline{|2n+1}}{2^{2n}\,\underline{|n}\,\underline{|n+1}}\left\{\cos \theta + \frac{n}{n+2} \cos 3\theta \right.$$

$$+ \frac{n(n-1)}{(n+2)(n+3)} \cos 5\theta + \frac{n(n-1)(n-2)}{(n+2)(n+3)(n+4)} \cos 7\theta$$

$$\left. + \ldots \right\};$$

and likewise

$$\sin^{2n}\theta = \frac{\underline{|2n}}{2^{2n-1}\,\underline{|n}\,\underline{|n}}\left\{\frac{1}{2} - \frac{n}{n+1}\cos 2\theta + \frac{n(n-1)}{(n+1)(n+2)}\cdot\right.$$
$$\left.\cos 4\theta - \frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)}\cdot\cos 6\theta + \ldots\right\}$$

and

$$\sin^{2n+1}\theta = \frac{\underline{|2n+1}}{2^{2n}\,\underline{|n}\,\underline{|n+1}}\left\{\sin\theta - \frac{n}{n+2}\sin 3\theta\right.$$
$$+ \frac{n(n-1)}{(n+2)(n+3)}\sin 5\theta - \frac{n(n-1)(n-2)}{(n+2)(n+3)(n+4)}\cdot$$
$$\left.\sin 7\theta + \ldots\right\}.$$

In an exactly similar manner it may be proved that

$$\cos^n\theta\cos n\theta = \frac{1}{2^n}\left\{1 + n\cos 2\theta + \frac{n(n-1)}{\underline{|2}}\cos 4\theta\right.$$
$$\left.+ \frac{n(n-1)(n-2)}{\underline{|3}}\cos 6\theta + \ldots\right\}$$

whether $n$ be even or odd; and in a similar manner like expressions can be obtained for $\cos^n\theta\sin n\theta$, $\sin^n\theta\cos n\theta$ and $\sin^n\theta\sin n\theta$, these expressions taking different forms in the latter two cases according as $n$ is odd or even.

*Problem III.*—Let us represent the displacement of an engine piston in terms of the stroke $2a$, the length of the connecting rod $l$, and the crank angle $\theta$. The expression giving the piston displacement, $x$, is

$$x = a\cos\theta + \sqrt{l^2 - a^2\sin^2\theta} - l \quad . \quad . \quad (1)$$

Let the ratio $a/l$ be denoted by $\varrho$

Then we have, by the Binomial Theorem,

$$x = a\cos\theta + \frac{a}{\varrho}\left\{1 - \frac{1}{2}\varrho^2\sin^2\theta - \frac{1}{8}\varrho^4\sin^4\theta\right.$$
$$\left.- \frac{1}{16}\varrho^6\sin^6\theta - \frac{5}{128}\varrho^8\sin^8\theta - \ldots - 1\right\}.$$

If we replace the powers of the sines by their expansions as found in Problem II, namely,

$$2 \sin^2 \theta = 1 - \cos 2\theta,$$
$$8 \sin^4 \theta = 3 - 4 \cos 2\theta + \cos 4\theta,$$
$$32 \sin^6 \theta = 10 - 15 \cos 2\theta + 6 \cos 4\theta - \cos 6\theta, \quad \text{etc.},$$

and arrange the result in the form of a Fourier's series, we get

$$\frac{x}{a} = -\left(\frac{\varrho}{4} + \frac{3\varrho^3}{64} + \frac{5\varrho^5}{256} + \dots\right) + \cos\theta$$
$$+ \left(\frac{\varrho}{4} + \frac{\varrho^3}{16} + \frac{15\varrho^5}{512} + \dots\right) \cos 2\theta$$
$$- \left(\frac{\varrho^3}{64} + \frac{3\varrho^5}{256} + \dots\right) \cos 4\theta + \left(\frac{\varrho^5}{512} + \dots\right) \cos 6\theta$$
$$- \dots$$

This expression is chiefly required in practice to obtain the acceleration of the piston. If $\theta = nt$, so that $n$ is the angular velocity of the crank, we have

$$\frac{d^2x}{dt^2} = n^2 a \left\{ - \cos\theta - \left(\varrho + \frac{\varrho^3}{4} + \frac{15\varrho^5}{128} \dots\right) \cos 2\theta \right.$$
$$\left. + \left(\frac{\varrho^3}{4} + \frac{3\varrho^5}{16} + \dots\right) \cos 4\theta - \left(\frac{9\varrho^5}{128} \dots\right) \cos 6\theta + \dots \right\}$$

The numerical maximum of this expression is when $\theta = 0$ and will be found to be $n^2 a(1 + \varrho)$ which can, of course, be obtained by differentiating (1), or the Binomial expansion, twice.

## The Direct Summation of Fourier's Series.

Closely allied with the preceding representation of an analytic periodic function as a Fourier series is the direct summing of a given Fourier series, which is simply the inverse process. How to tell whether a given Fourier series represents an analytic function

or an artificial one will be dealt with in Chapter VI. A single example will suffice to illustrate the method adopted in either case.

*Example I.*—Find the sum of the two series

$$c \cos\theta + \frac{c^2}{2}\cos 2\theta + \frac{c^3}{3}\cos 3\theta + \dots$$

and $$c \sin\theta + \frac{c^2}{2}\sin 2\theta + \frac{c^3}{3}\sin 3\theta + \dots$$

where $$|c| < 1.$$

If we denote the sums of these series by C and S respectively, we have, on multiplying the second one by $i$ and adding it to the first,

$$C + iS = ce^{i\theta} + \frac{c^2}{2}e^{2i\theta} + \frac{c^3}{3}e^{3i\theta} + \dots$$

which is easily identified as $-\log(1 - ce^{i\theta})$.

Taking exponentials of both sides after changing their signs, we have

$$e^{-C}(\cos S - i\sin S) = 1 - ce^{i\theta},$$

whence

$$e^{-C}\cos S = 1 - c\cos\theta \quad \text{and} \quad e^{-C}\sin S = c\sin\theta$$

and hence $$\tan S = \frac{c\sin\theta}{1 - c\cos\theta}$$

and

$$e^{-2C}(\cos^2 S + \sin^2 S) = e^{-2C} = 1 - 2c\cos\theta + c^2$$

and so

$$C = -\tfrac{1}{2}\log(1 - 2c\cos\theta + c^2)$$

and $$S = \tan^{-1}\frac{c\sin\theta}{1 - c\cos\theta}.$$

If $c = 1$, these simplify to

$$C = \log(\tfrac{1}{2}\operatorname{cosec}\theta/2) \qquad (\text{from } \theta = 0 \text{ to } \pi)$$

and $$S = \frac{\pi}{2} - \frac{\theta}{2} \qquad (\text{from } \theta = 0 \text{ to } \pi),$$

the latter agreeing with the result in (7) p. 42.

## EXAMPLES.

1. The length of the focal radius vector of an ellipse of eccentricity $e$ and semi-latus-rectum $l$ is given by

$$r = \frac{l}{1 - e\cos\theta}.$$

Prove, by comparing this expression with that in Problem I, that the Fourier's series for $r$ is

$$r = l\sec\beta\{1 + 2\tan\beta/2\cos\theta + 2\tan^2\beta/2\cos 2\theta + 2\tan^3\beta/2\cos 3\theta + \ldots\}$$

where $\sin\beta = e$. Note that if $a$ and $b$ are the semi major and minor axes respectively, we have $b = a\cos\beta$ and, since $l = b^2/a$, the factor $l\sec\beta$ is simply equal to $b$.

2. Prove that the sums of the series

$$1 + \frac{c}{\lfloor 1}\cos\theta + \frac{c^2}{\lfloor 2}\cos 2\theta + \frac{c^3}{\lfloor 3}\cos 3\theta + \ldots$$

and

$$\frac{c}{\lfloor 1}\sin\theta + \frac{c^2}{\lfloor 2}\sin 2\theta + \frac{c^3}{\lfloor 3}\sin 3\theta + \ldots$$

are $$e^{c\cos\theta}\cos(c\sin\theta)$$

and $$e^{c\cos\theta}\sin(c\sin\theta)$$

respectively.

3. Show that the sums of the series

$$\cos\theta + n\cos 3\theta + \frac{n(n-1)}{\lfloor 2}\cos 5\theta + \ldots$$

and $$\sin\theta + n\sin 3\theta + \frac{n(n-1)}{\lfloor 2}\sin 5\theta + \ldots$$

are

$$2^n\cos^n\theta\cos(n+1)\theta$$

and $$2^n\cos^n\theta\sin(n+1)\theta$$

respectively.

4. Show that the sums of the series

$$1 + c\cos\theta + c^2\cos 2\theta + c^3\cos 3\theta + \ldots$$

and $$c\sin\theta + c^2\sin 2\theta + c^3\sin 3\theta + \ldots$$

where $|c|$ is less than unity are

$$\frac{1 - c\cos\theta}{1 - 2c\cos\theta + c^2} \quad\text{and}\quad \frac{c\sin\theta}{1 - 2c\cos\theta + c^2}$$

respectively.

5. Obtain the results:

$$\cos\theta + \frac{\cos 2\theta}{2} + \frac{\cos 3\theta}{3} + \ldots = \log\left|\tfrac{1}{2}\operatorname{cosec}\theta/2\right|$$

$$\cos\theta - \frac{\cos 2\theta}{2} + \frac{\cos 3\theta}{3} - \ldots = -\log\left|\tfrac{1}{2}\sec\theta/2\right|$$

$$\cos\theta + \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} + \ldots = \tfrac{1}{2}\log\left|\cot\theta/2\right|$$

$$\sin\theta - \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} - \ldots = \tfrac{1}{2}\log\left|\tan(\theta/2 + \pi/4)\right|$$

$$= \tfrac{1}{2}\log\left|\sec\theta + \tan\theta\right|$$

These results should be compared with the series

$$\sin\theta + \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} + \ldots$$

and

$$\sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \ldots$$

which represent the function in Fig. 12, with the origins at the points marked 0 and $\pi$ respectively; and also with the series

$$\sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \ldots$$

and

$$\cos\theta - \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} - \ldots$$

which represent the function in Fig. 9(*a*), where the origin is taken at the point A, and midway between A and D, respectively.

# CHAPTER V

## ON THE REPRESENTATION OF EMPIRICAL FUNCTIONS BY FOURIER'S SERIES

THE empirical function which we desire to represent by a Fourier's series is necessarily given us either: (*a*) by some experimentally obtained graph; or (*b*) by some experimentally obtained set of points. In either case, the ideal results that were aimed at have only been reached to a certain degree of approximation. This limited accuracy of experimental results should always be remembered when subjecting them to mathematical operations.

When a graph is given, two courses are open to us: we may either apply some form of mechanical contrivance called a Harmonic Analyser to the graph to evaluate the Fourier Constants, or we may select a finite number of points on the graph and proceed as if these points were all that we knew about it.

Several mechanical contrivances for the harmonic analysis of graphs have been invented which possess very various limitations with respect to the number of harmonics they are capable of dealing with; sometimes, indeed, too severe to make the apparatus of much practical value. Space prohibits us from dealing with these machines here, since nothing but a complete explanatory description with diagrams and illustrations would be of any value. A useful account of several such machines will be found in the Article on Calculating Machines in the 11th Edition of the *Encyclopædia Britannica*. References to the original

description of several of these machines will be found in this Article.

These machines are all both complicated and expensive, and, apart from the mechanical errors of the machine (and sometimes theoretical errors too), the accuracy of the results depends upon the accuracy with which one can move the pointer of the machine along the given graph. We do not consider these machines essential for practical use ; and a new method of harmonic analysis which is described below, we think, renders them all the more dispensable except perhaps where a very large number of similar graphs have to be analysed for the same harmonic.

We will suppose, therefore, in the remainder of this chapter, that we are given a set of points on a periodic function, which we will suppose to be situated on equidistant ordinates, and that we require to obtain a Fourier's series which will pass through the given points. There are two distinct ways in which this problem may be solved. In the first method we find the Fourier's series with the least number of the lowest harmonics which will give a function passing through the given points. This is the usual method of harmonic analysis and will be explained in detail.

The new method referred to, which has not, as far as we know, been published before, is quite distinct and consists in joining the given points by straight lines (or by arcs of parabolas of the second or third degree) and then obtaining the Fourier's series representing this built-up curve from the discontinuities of the function defined by this series of arcs. This method is somewhat more laborious than the first one, but it has the advantage that we know exactly what the resulting series represents. There is, however, no utility in applying this method when *all* that is known is a set of ordinates of the given function; for we could not then tell which of two

different curves passing *through* the given points represented the function most accurately *between* the points. But the method is valuable when a graph of the function is given if we take ordinates at such a distance apart that it is obvious to the eye that the form of the curve between two ordinates is very approximately parabolic. In this way we may obtain a Fourier's series which represents the given graph with much greater accuracy than does the simplest Fourier series passing through the given points. Since points can be measured on a graph much more accurately than the pointer of an instrument can be moved over the graph by hand; and since, also, with a reasonable number of ordinates, the various parabolic arcs will agree more accurately (through their entire length) with the given graph than a pointer moved over it by hand however carefully one does it; we consider this method of harmonic analysis as being more accurate than any Harmonic Analyser.

*The Simplest Fourier's Series passing through 2p points per period.*

We will suppose that the period is $2\pi$ as usual and that we require to find the curve with the lowest possible harmonics which passes through the $2p$ points

$$\left(\frac{\pi}{p}, A_1\right), \left(\frac{2\pi}{p}, A_2\right) \ldots (2\pi, A_{2p}).$$

We are at liberty to assume any series with $2p$ unknown coefficients. If we assume

$$f(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + \ldots + a_p \cos pt$$
$$+ b_1 \sin t + b_2 \sin 2t + \ldots + b_p \sin pt,$$

we have $2p + 1$ unknowns; more than can be determined, but we note that when $t = N\frac{\pi}{p}$, $pt = N\pi$ and

so the last sine term is seen to be zero at each ordinate: it therefore makes no difference to the curve passing *through* the given points, so we omit it. We will assume, then, the expression

$$f(t) = \frac{a_0}{2} + a_1 \cos t + \ldots + a_{p-1} \cos (p-1)t + \frac{a_p}{2} \cos pt$$

$$+ b_1 \sin t + \ldots + b_{p-1} \sin (p-1)t \quad (1)$$

the reason for writing $\frac{a_0}{2}$ and $\frac{a_p}{2}$ instead of $a_0$ and $a_p$ is, as will be seen later, so that all the a's can be obtained from the same expression.

If in this expression we put $t = \frac{\pi}{p}, \frac{2\pi}{p}, \ldots 2\pi$ successively the left-hand side becomes $A_1, A_2, \ldots A_{2p}$ respectively by hypothesis. If we then add the $2p$ equations so obtained all terms cancel save the first by (F) and (G), p. 9, and so we have

$$\frac{2p}{2}a_0 = \Sigma A_q.$$

If we again write $t = \frac{\pi}{p}, \frac{2\pi}{p} \ldots 2\pi$ successively in (1); but before adding the equations multiply them by $\cos \frac{n\pi}{p}, \cos \frac{2n\pi}{p}, \ldots \cos \frac{2pn\pi}{p}$ respectively, where $n \ngtr p$ and then add, we can see by equations (H) to (J), p. 10, that all the cosine terms will vanish save $a_n$, and that all the sine terms will vanish also; and hence we have

$$a_n \Sigma \cos^2 \frac{qn\pi}{p} = \Sigma A_q \cos \frac{qn\pi}{p}.$$

By the use of the formula $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$, we easily see that the value of the sum on the left-hand side is $p$, since there are $2p$ terms, except in the case

when $n = p$ or $n = 0$, when the sum is $2p$, but since we have allowed for this by writing $\frac{\mathrm{a}_0}{2}$ and $\frac{\mathrm{a}_p}{2}$ instead of $\mathrm{a}_0$ and $\mathrm{a}_p$, we see that

$$\mathrm{a}_n = \frac{1}{p}\Sigma \mathrm{A}_q \cos \frac{qn\pi}{p} \quad . \quad . \quad . \quad . \quad (2)$$

in all cases.

In an exactly similar manner, we find

$$\mathrm{b}_n = \frac{1}{p}\Sigma \mathrm{A}_q \sin \frac{qn\pi}{p} \quad . \quad . \quad . \quad . \quad (3)$$

We have therefore solved the given problem.

We will now prove that, as the number of ordinates tends to infinity, the constants $\mathrm{a}_n$ and $\mathrm{b}_n$ tend to the Fourier constants $a_n$ and $b_n$.

$\mathrm{A}_q$ is the ordinate at $t = \frac{q\pi}{p}$: we may denote it by $\mathrm{A}_t$ instead. In this case the factor $\cos \frac{qn\pi}{p}$ must be denoted by $\cos nt$. The factor $\frac{1}{p}$ is very small, when $p$ is large, and is $\frac{1}{\pi}$ of the distance between successive ordinates; but this distance, in the limit, is $dt$, and hence $\frac{1}{p}$ becomes $\frac{dt}{\pi}$ in the limit when the summation becomes an integration. The summation extends from $t = \frac{\pi}{p}$ to $t = 2\pi$ which, in the limit, is from $0$ to $2\pi$. Hence

$$\frac{1}{p}\,\Sigma\,\mathrm{A}_q \cos \frac{qn\pi}{p} \quad \text{becomes} \quad \frac{1}{\pi}\int_0^{2\pi} \mathrm{A}_t \cos nt\, dt,$$

which is the value of the Fourier constant $a_n$. Similarly, $\mathrm{b}_n$ becomes $b_n$.

The expressions in (2) and (3) which consist of $2p$ terms can be reduced to one-quarter of the number of terms in the case of the four special classes of periodic functions. In what follows we shall denote the distance $\pi/p$ between two consecutive ordinates by $\alpha$ and by $A_q$ we shall always mean the ordinate at $t = q\alpha$.

For an odd function, let us divide the half-wave into $p$ parts and let the ordinates at $0, \alpha, 2\alpha, \ldots$ be $0, B_1, B_2 \ldots B_{p-1}, 0$. The complete set of ordinates over a whole period is thus

$$0, B_1, B_2 \ldots B_{p-1}, 0, -B_{p-1}, \ldots -B_1, 0.$$

If these be inserted respectively for the A's in the formula $b_n = \frac{1}{p}\Sigma A_q \sin qn\alpha$, and if we note that the sine factors for $B_1$ and $-B_1$ are numerically equal but of opposite sign; and that the sine factors for $B_1$ and $B_{p-1}$ are also numerically equal, and of the *same* sign when $n$ is odd and *opposite* when $n$ is even, we see that

$$\begin{aligned} b_n = \frac{2}{p}\Big\{&\Big(B_1 + (-1)^{n-1} B_{p-1}\Big) \sin n\alpha \\ &+ \Big(B_2 + (-1)^{n-1} B_{p-2}\Big) \sin 2n\alpha \\ &+ \Big(B_3 + (-1)^{n-1} B_{p-3}\Big) \sin 3n\alpha + \ldots\Big\} \end{aligned} \quad \ldots \quad (4)$$

the series continuing till all the B's have been taken once only. If there are an odd number of B's, *i.e.*, if $p$ is even, the last term in this expression will be $B_{p/2} \sin \frac{n\pi}{2}$. The assumed series for the function is, of course,

$$f(t) = b_1 \sin t + b_2 \sin 2t + \ldots + b_{p-1} \sin (p-1) t. \quad \ldots \quad (5)$$

For an even function, let the ordinates be $C_0$, $C_1$, $C_2 \ldots C_p$ at $t = 0, \alpha, 2\alpha \ldots \pi$; so that the ordinates over a whole period may be denoted by $C_0, C_1 \ldots C_{p-1}, C_p, C_{p-1} \ldots C_1, C_0$. Inserting these for the A's in $a_n = \frac{1}{p}\Sigma A_q \cos nq\alpha$, and grouping together, as before, the four terms which have the cosine factor of the same numerical magnitude, we get

$$a_n = \frac{2}{p}\left\{\frac{C_0 + (-1)^n C_p}{2} + \left(C_1 + (-1)^n C_{p-1}\right)\cos n\alpha \right.$$
$$\left. + \left(C_2 + (-1)^n C_{p-2}\right)\cos 2n\alpha + \ldots\right\} \quad . \quad . \quad (6)$$

the series continuing till all the C's have been taken once only. The series in this case is

$$f(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + \ldots$$
$$+ a_{p-1}\cos(p-1)t + \frac{a_p}{2}\cos pt \quad . \quad . \quad . \quad (7)$$

For an odd harmonic function, let us take the origin at one of the points where the curve cuts the axis, which it necessarily does at intervals of $\pi$. Let the ordinates over a half period be denoted by $0, D_1, D_2, \ldots D_{p-1}, 0$, so that the complete set of ordinates over a period are given by

$$0, D_1, D_2, \ldots D_{p-1}, 0, -D_1, -D_2 \ldots -D_{p-1}.$$

Substituting these for the A's in (2) and (3) we easily get, on remembering that $n$ is odd,

$$a_n = \frac{2}{p}\left\{\left(D_1 - D_{p-1}\right)\cos n\alpha + \left(D_2 - D_{p-2}\right)\cos 2n\alpha \right.$$
$$\left. + \left(D_3 - D_{p-3}\right)\cos 3n\alpha + \ldots\right\} \quad . \quad . \quad (8)$$

and

$$b_n = \frac{2}{p}\left\{\left(D_1 + D_{p-1}\right)\sin n\alpha + \left(D_2 + D_{p-2}\right)\sin 2n\alpha + \left(D_3 + D_{p-3}\right)\sin 3n\alpha + \ldots\right\} \quad . \quad . \quad (9)$$

the series being continued till all the D's have been taken once only. It must be noted that these expressions *are not to be worked out* for even values of $n$.

The series representing the function is

$$f(t) = a_1 \cos t + a_3 \cos 3t + a_5 \cos 5t + \ldots + b_1 \sin t + b_3 \sin 3t + \ldots \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If $p$ is even the cosines stop with the term $a_{p-1} \cos (p - 1)t$ and if $p$ is odd with $\frac{a_p}{2} \cos pt$, and the sine terms stop with $b_{p-1} \sin (p-1)t$ or $b_{p-2} \sin (p-2)t$ respectively. It can easily be seen that the total number of terms in both cases is $p$. It may be noted that since $f(0) = 0$ the a's satisfy the condition that

$$a_1 + a_3 + a_5 + \ldots = 0.$$

For a Class IV function, we need only measure ordinates over a quarter of a period. Let us divide the quarter period into $p/2$ parts, $p$ being assumed even, by ordinates at $\frac{\pi}{p} = \alpha$ apart, as before.

For an odd Class IV function, let the ordinates be $0, E_1, E_2, \ldots E_{p/2}$ at $0, \alpha, 2\alpha, \ldots \frac{\pi}{2}$. Then from (9) we have

$$b_n = \frac{2}{p}\left\{2E_1 \sin n\alpha + 2E_2 \sin 2n\alpha + \ldots + E_{p/2} \sin n\frac{\pi}{2}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

G

the factor 2 inside the brackets being associated with all the terms except the last. The series is

$$f(t) = b_1 \sin t + b_3 \sin 3t + \ldots + b_{p-1} \sin (p-1)t \qquad \ldots \quad (12)$$

For an even Class IV function, let $p$ again be even and let the ordinates over a quarter period be $F_0$, $F_1$, . . . $F_{p/2-1}$, 0. Then from (6), since now,

$$C_p = -C_0 = -F_0; \; C_{p-1} = -C_1 = -F_1, \text{ etc.,}$$

we have

$$a_n = \frac{2}{p}\left\{F_0 + 2F_1 \cos n\alpha + 2F_2 \cos 2n\alpha + \ldots\right\} \qquad \ldots \quad (13)$$

the series being

$$f(t) = a_1 \cos t + a_3 \cos 3t + \ldots + a_{p-1} \cos (p-1)t \qquad \ldots \quad (14)$$

The student must remember that formulæ (11) and (13) are only to be worked out for odd values of $n$.

There is still another great simplification which may be made in the practical calculation of these coefficients. Let us change $n$ into $p - n$ in any of the expressions for $a_n$ or $b_n$. Since $qn\alpha$ i.e. $qn\pi/p$ then becomes $q(p-n)\pi/p$ or $q\pi - qn\alpha$ we see that a cosine term will be unaltered if $q$ is even, and merely have its sign changed if $q$ is odd, and *vice versa* with a sine term. So that the series for $a_{p-n}$ or $b_{p-n}$ is always the same as that for $a_n$ or $b_n$ with the signs of alternate terms changed. Hence we add up the odd and even terms separately; and the sum of these two sums gives $a_n$ or $b_n$ and the difference gives $a_{p-n}$ or $b_{p-n}$. In order to avail of this simplification in the case of odd harmonic functions, it is obvious that $p$, the number of ordinates per half-period, must be even.

*Schedules for Harmonic Analysis.*

We are now in a position to construct Schedules for Harmonic Analysis. Such a Schedule for a Class I function when using 10 ordinates per half-period is shown on page 84. The given ordinates are denoted by the $y$'s; both $y_0$ and $y_{10}$ being necessarily zero. The sums and differences of these when arranged as shown are denoted by the U's and V's respectively. The first column contains the numbers from 0 to 5, and the second contains the value of the factor $\frac{2}{p}\sin\frac{n\pi}{p}$, when $p = 10$, for the values of $n$ from 0 to 5. Nine further columns for the nine harmonics are taken, arranged and headed as shown. We then multiply the U's in turn by the successive sine factors and place the products, which are denoted by the $u$'s, in the columns for the 1st and 9th harmonic; placing them alternately in the third and fourth columns as shown. If the sums of these columns by denoted by $P_1$ and $Q_1$ we have $P_1 + Q_1 = b_1$ and $P_1 - Q_1 = b_9$.

In filling in the $u$'s for the 3rd and 7th harmonics we place $u_1$ in column 5 in line 3; $u_2$ should go in column 6, line 6, but the sine factor of this line is the same as for line 4, so we place it in this line; $u_3$ should go in column 5, line 9, but line 1 gives the correct sine factor, so it is placed there; $u_4$, which should be in line 12, has the same sine factor as line 2, with its sign changed, so it is placed as shown. The directions for placing these $u$'s may be stated as follows: starting with $u_1$ in line 3 go up and down the lines from 0 to 5 in steps of 3 at a time placing the $u$'s alternately in columns 5 and 6, and changing the sign of the term each time the sine factor passes through zero. For the 5th harmonic, $u_1$, $u_3$, and $u_5$ occupy line 5, while $u_2$ and $u_4$ occupy line 0; but since the sine factor of

SCHEDULE I.—FOR ANALYSIS OF SINE HARMONIC OR CLASS I FUNCTIONS.
(10 ORDINATES PER HALF-PERIOD.)

| | | | | | | |
|---|---|---|---|---|---|---|
| Ordinates | o | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
| | o | $y_9$ | $y_8$ | $y_7$ | $y_6$ | |
| Sum ...... | o | $U_1$ | $U_2$ | $U_3$ | $U_4$ | $U_5$ |
| Difference | o | $V_1$ | $V_2$ | $V_3$ | $V_4$ | $V_5$ |

| Line. | Sine factor. | Harmonic. 1st (& 9th). | | 3rd (& 7th). | | 5th. | 2nd (& 8th). | | 4th (& 6th). | |
|---|---|---|---|---|---|---|---|---|---|---|
| o | o | | | | | o | o | | o | |
| 1 | ·0618 | $u_1$ | | $u_3$ | | | | | | |
| 2 | ·1176 | | $u_2$ | | $-u_4$ | | $v_1$ | $v_4$ | $-v_3$ | $v_2$ |
| 3 | ·1618 | $u_3$ | | $u_1$ | | | | | | |
| 4 | ·1902 | | $u_4$ | | $u_2$ | | $v_3$ | $v_2$ | $v_1$ | $-v_4$ |
| 5 | ·2000 | $u_5$ | | $-u_5$ | | $\left.\begin{matrix} u_1 \\ -u_3 \\ u_5 \end{matrix}\right\}$ | | | | |
| Sum | | $P_1$ | $Q_1$ | $P_3$ | $Q_3$ | $P_5$ | $P_2$ | $Q_2$ | $P_4$ | $Q_4$ |
| | | $P_1 + Q_1 = b_1$<br>$P_1 - Q_1 = b_9$ | | $P_3 + Q_3 = b_3$<br>$P_3 - Q_3 = b_7$ | | $P_5 = b_5$ | $P_2 + Q_2 = b_2$<br>$P_2 - Q_2 = b_8$ | | $P_4 + Q_4 = b_4$<br>$P_4 - Q_4 = b_6$ | |

The $u$'s and $v$'s are the U's and V's multiplied by the sine factor of the line in which they are situated.

SCHEDULE I.—FOR ANALYSIS OF SINE HARMONIC FUNCTIONS.
(10 ORDINATES PER HALF-PERIOD.)

| | | | | | | |
|---|---|---|---|---|---|---|
| Ordinates | 0 | 6 | 12 | 18 | 24 | 20 |
| | 0 | 4 | 8 | 12 | 16 | |
| Sum ...... | 0 | 10 | 20 | 30 | 40 | 20 |
| Difference | 0 | 2 | 4 | 6 | 8 | 20 |

| Line. | Sine factor. | Harmonic. | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 1st (& 9th). | | 3rd (& 7th). | | 5th. | 2nd (& 8th). | | 4th (& 6th). | |
| 0 | 0 | | | | | | | | | |
| 1 | ·0618 | ·618 | | 1·854 | | | | | | |
| 2 | ·1176 | | 2·352 | | −4·704 | | ·235 | ·941 | −·706 | ·470 |
| 3 | ·1618 | 4·854 | | 1·618 | | | | | | |
| 4 | ·1902 | | 7·608 | | 3·804 | | 1·141 | ·761 | ·380 | −1·522 |
| 5 | ·2000 | 4·000 | | −4·000 | | 2·000<br>−6·000<br>+4·000 | | | | |
| Sum | | 9·472 | 9·960 | − ·528 | − ·900 | 0 | 1·376 | 1·702 | −·326 | −1·052 |
| | | $b_1 =$ 19·432<br>$b_9 = -$ ·488 | | $b_3 = -$ 1·428<br>$b_7 = +$ ·372 | | $b_5 = 0$ | $b_2 =$ 3·078<br>$b_8 = -$ ·326 | | $b_4 = -$ 1·378<br>$b_6 = +$ ·726 | |

SCHEDULE II.—FOR ANALYSIS OF COSINE HARMONIC OR CLASS II FUNCTIONS.
(10 ORDINATES PER HALF-PERIOD.)

$$\begin{array}{lcccccc}
\text{Ordinates} & y_0 & y_1 & y_2 & y_3 & y_4 & y_5 \\
 & y_{10} & y_9 & y_8 & y_7 & y_6 & \\
\hline
\text{Sum} \ldots\ldots & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 \\
\text{Difference} & V_0 & V_1 & V_2 & V_3 & V_4 & V_5
\end{array}$$

| Line. | Cosine factor. | Harmonic. 0th (& 10th). | | 2nd (& 8th). | | 4th (& 6th). | | 1st (& 9th). | | 3rd (& 7th). | | 5th. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | ·2000 | $u_0/2$, $u_2$, $u_4$ | $u_1$, $u_3$, $u_5$ | $\frac{u_0}{2}$ | $-u_5$ | $\frac{u_0}{2}$ | $+u_5$ | $\frac{v_0}{2}$ | | $\frac{v_0}{2}$ | | $v_0/2$, $-v_2$, $+v_4$ |
| 1 | ·1902 | | | | | | | | $v_1$ | | $-v_3$ | |
| 2 | ·1618 | | | $-u_4$ | $u_1$ | $-u_2$ | $-u_3$ | $v_2$ | | $-v_4$ | | |
| 3 | ·1176 | | | | | | | | $v_3$ | | $v_1$ | |
| 4 | ·0618 | | | $u_2$ | $-u_3$ | $+u_4$ | $u_1$ | $v_4$ | | $-v_2$ | | |
| 5 | 0 | | | | | | | | 0 | | 0 | 0 |
| Sum | | $P_0$ | $Q_0$ | $P_2$ | $Q_2$ | $P_4$ | $Q_4$ | $P_1$ | $Q_1$ | $P_3$ | $Q_3$ | $P_5$ |
| | | $\frac{P_0+Q_0}{2}=\frac{a_0}{2}$ | | $P_2+Q_2=a_2$ | | $P_4+Q_4=a_4$ | | $P_1+Q_1=a_1$ | | $P_3+Q_3=a_3$ | | $P_5=a_5$ |
| | | $\frac{P_0-Q_0}{2}=\frac{a_{10}}{2}$ | | $P_2-Q_2=a_8$ | | $P_4-Q_4=a_6$ | | $P_1-Q_1=a_9$ | | $P_3-Q_3=a_7$ | | |

The $u$'s and the $v$'s are the U's and V's multiplied by the cosine factor of the line in which they are situated.

SCHEDULE II.—FOR ANALYSIS OF COSINE HARMONIC FUNCTIONS.
(10 ORDINATES PER HALF-PERIOD.)

| | | | | | | |
|---|---|---|---|---|---|---|
| Ordinates | 10 | 15 | 20 | 25 | 30 | 25 |
| | 0 | 5 | 10 | 15 | 20 | |
| Sum ...... | 10 | 20 | 30 | 40 | 50 | 25 |
| Difference | 10 | 10 | 10 | 10 | 10 | 25 |

| Line. | Cosine factor. | Harmonic. | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0th (& 10th). | | 2nd (& 8th). | | 4th (& 6th). | | 1st (& 9th). | | 3rd (& 7th). | | 5th. |
| 0 | ·2000 | 1·0<br>6·0<br>10·0 | 4·0<br>8·0<br>5·0 | 1·0 | −5·0 | 1·0 | 5·0 | 1·0 | | 1·0 | | 1·0<br>−2·0<br>+2·0 |
| 1 | ·1902 | | | | | | | | 1·902 | | −1·902 | |
| 2 | ·1618 | | | −8·09 | 3·24 | −4·85 | −6·47 | 1·618 | | −1·618 | | |
| 3 | ·1176 | | | | | | | | 1·176 | | −1·176 | |
| 4 | ·0618 | | | 1·85 | −2·47 | 3·09 | 1·24 | ·618 | | − ·618 | | |
| 5 | 0 | | | | | | | | | | | |
| Sum | | 17 | 17 | −5·24 | −4·23 | − ·76 | −23 | 3·236 | 3·078 | −1·236 | − ·726 | 1·0 |
| | | $\frac{a_0}{2} = 17$<br>$\frac{a_{10}}{2} = 0$ | | $a_2 = -9·47$<br>$a_8 = -1·01$ | | $a_4 = -·99$<br>$a_6 = -·53$ | | $a_1 = 6·314$<br>$a_9 = ·158$ | | $a_3 = -1·962$<br>$a_7 = -·510$ | | $a_5 = 1·0$ |

SCHEDULE III.—FOR ANALYSIS OF ODD HARMONIC OR CLASS III FUNCTIONS. (14 ORDINATES PER HALF-PERIOD.)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Ordinates | 0 | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | $y_7$ |
| | 0 | $y_{13}$ | $y_{12}$ | $y_{11}$ | $y_{10}$ | $y_9$ | $y_8$ | |
| Sum ...... | 0 | $U_1$ | $U_2$ | $U_3$ | $U_4$ | $U_5$ | $U_6$ | $U_7$ |
| Difference | 0 | $V_1$ | $V_2$ | $V_3$ | $V_4$ | $V_5$ | $V_6$ | $V_7$ |

| Line. | Sine factor. | Cosine factor. | Harmonic. 1st (& 13th). | | 3rd (& 11th). | | 5th (& 9th). | | 7th. |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | *·1429* | | | | | | | $\begin{cases}-v_2\\+v_4\\-v_6\end{cases}$ |
| 1 | ·0318 | *·1392* | $u_1$ | $v_1$ | $-u_5$ | $-v_5$ | $-u_3$ | $-v_3$ | |
| 2 | ·0620 | *·1287* | $v_2$ | $u_2$ | $v_4$ | $u_4$ | $+v_6$ | $+u_6$ | |
| 3 | ·0891 | *·1117* | $u_3$ | $v_3$ | $u_1$ | $v_1$ | $-u_5$ | $-v_5$ | |
| 4 | ·1117 | *·0891* | $v_4$ | $u_4$ | $-v_6$ | $-u_6$ | $v_2$ | $u_2$ | |
| 5 | ·1287 | *·0620* | $u_5$ | $v_5$ | $u_3$ | $v_3$ | $u_1$ | $v_1$ | |
| 6 | ·1392 | *·0318* | $v_6$ | $u_6$ | $v_2$ | $u_2$ | $-v_4$ | $-u_4$ | |
| 7 | ·1429 | 0 | $u_7$ | 0 | $-u_7$ | 0 | $u_7$ | 0 | $u_1-u_3+u_5-u_7$ |
| Sum of $u$'s<br>Sum of $v$'s | | | $P_1$<br>$R_1$ | $Q_1$<br>$S_1$ | $P_3$<br>$R_3$ | $Q_3$<br>$S_3$ | $P_5$<br>$R_5$ | $Q_5$<br>$S_5$ | $P_7$<br>$R_7$ |
| | | | $P_1+Q_1=b_1$<br>$P_1-Q_1=b_{13}$<br>$R_1+S_1=a_1$<br>$R_1-S_1=a_{13}$ | | $P_3+Q_3=b_3$<br>$P_3-Q_3=b_{11}$<br>$R_3+S_3=a_3$<br>$R_3-S_3=a_{11}$ | | $P_5+Q_5=b_5$<br>$P_5-Q_5=b_9$<br>$R_5+S_5=a_5$<br>$R_5-S_5=a_9$ | | $P_7=b_7$<br>$R_7=a_7$ |

The $u$'s are the U's multiplied by the sine factor, and the $v$'s are the V's multiplied by the cosine factor, of the line on which they are situated.

## SCHEDULE III.—FOR ANALYSIS OF ODD HARMONIC FUNCTIONS. (14 ORDINATES PER HALF-PERIOD.)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Ordinates | 0 | 104 | 192 | 264 | 320 | 360 | 384 | 392 |
| | 0 | 104 | 192 | 264 | 320 | 360 | 384 | |
| Sum ...... | 0 | 208 | 384 | 528 | 640 | 720 | 768 | 392 |
| Difference | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 392 |

| Sine factor. | Cosine factor. | Harmonic. 1st (& 13th). | | 3rd (& 11th). | | 5th (& 9th). | | 7th. |
|---|---|---|---|---|---|---|---|---|
| 0 | ·1429 | | | | | | | |
| ·0318 | ·1392 | 6·6 | | −22·9 | | −16·8 | | |
| ·0620 | ·1287 | | 23·8 | | 39·7 | | 47·6 | |
| ·0891 | ·1117 | 47·0 | | 18·5 | | −64·1 | | |
| ·1117 | ·0891 | | 71·5 | | −85·8 | | 42·9 | |
| ·1287 | ·0620 | 92·8 | | 68·0 | | 26·8 | | |
| ·1392 | ·0318 | | 106·9 | | 53·4 | | −89·2 | |
| ·1429 | 0 | 56·0 | | −56·0 | | 56·0 | | 29·7 − 70·4<br>102·9 − 5·6 |
| Sum of "sums."<br>Sum of "diffs." | | 202·4 | 202·2 | 7·6 | 7·3 | 1·9 | 1·3 | |
| | | $b_1 = 404·6$<br>$b_{13} = ·2$<br>$a_1 = 0$<br>$a_{13} = 0$ | | $b_3 = 14·9$<br>$b_{11} = ·3$<br>$a_3 = 0$<br>$a_{11} = 0$ | | $b_5 = 3·2$<br>$b_9 = ·6$<br>$a_5 = 0$<br>$a_9 = 0$ | | $b_7 = 1·2$<br>$a_7 = 0$ |

TABLE OF SINE OR COSINE FACTORS FOR HARMONIC ANALYSIS.

| *p* or Number of Ordinates per Half-period. | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ·353 | ·167 | ·096 | ·0618 | ·0431 | ·0318 | ·0244 | ·0193 | ·0156 | ·0129 | ·0109 |
| ·500 | ·289 | ·177 | ·1176 | ·0833 | ·0620 | ·0478 | ·0380 | ·0309 | ·0256 | ·0216 |
| | ·333 | ·231 | ·1618 | ·1179 | ·0891 | ·0695 | ·0555 | ·0454 | ·0378 | ·0319 |
| | | ·250 | ·1902 | ·1443 | ·1117 | ·0884 | ·0714 | ·0588 | ·0492 | ·0417 |
| | | | ·2000 | ·1610 | ·1287 | ·1039 | ·0851 | ·0707 | ·0595 | ·0507 |
| | | | | ·1667 | ·1392 | ·1155 | ·0962 | ·0809 | ·0687 | ·0589 |
| | | | | | ·1429 | ·1226 | ·1044 | ·0891 | ·0765 | ·0661 |
| | | | | | | ·1250 | ·1094 | ·0951 | ·0827 | ·0722 |
| | | | | | | | ·1111 | ·0988 | ·0872 | ·0770 |
| | | | | | | | | ·1000 | ·0900 | ·0805 |
| | | | | | | | | | ·0909 | ·0826 |
| | | | | | | | | | | ·0833 |

this line is zero, they are omitted. The even harmonics are calculated from the V's, but the arrangement of the *v*'s, which are the V's multiplied by the sine factor of the line on which they are situated, being exactly similar to that for the *u*'s, need not be further explained.

In an exactly similar manner, a Schedule for the analysis of any odd function using any number of ordinates per half-period can easily be constructed. A table of the sine factors required for a Schedule for any even number of ordinates from 4 to 24 per half-period is given above.

Page 85 shows Schedule I worked out for a particular example. The equation of the resulting Fourier's series there obtained is

$$f(t) = 19{\cdot}43 \sin t + 3{\cdot}08 \sin 2t - 1{\cdot}43 \sin 3t - 1{\cdot}38 \sin 4t + {\cdot}73 \sin 6t + {\cdot}37 \sin 7t - {\cdot}33 \sin 8t - {\cdot}49 \sin 9t.$$

The given points will be found to lie on two straight lines and the Fourier analysis of this function is

$$f(t) = \frac{200}{\pi^2}\left\{\sin\frac{2\pi}{5}\,\frac{\sin t}{1^2} + \sin\frac{4\pi}{5}\,\frac{\sin 2t}{2^2} + \ldots\right\}$$

or

$$f(t) = 19{\cdot}27 \sin t + 2{\cdot}98 \sin 2t - 1{\cdot}32 \sin 3t - 1{\cdot}20 \sin 4t + {\cdot}53 \sin 6t + {\cdot}24 \sin 7t - {\cdot}18 \sin 8t - {\cdot}24 \sin 9t + \ldots$$

with which the Harmonic Analysis is in very good agreement especially for the lower harmonics.

Page 86 gives the Schedule required for a Class II function, using 10 ordinates per half-period. It is constructed similar to Schedule I and is an embodiment of formula (6), just as Schedule I is an embodiment of formula (4). Page 87 shows the working out of a concrete example. It will be seen that the given points lie on two straight lines. The Fourier analysis of this function is easily obtained from its discontinuities. It has four discontinuities, of slope only, at $0, \pi$ and $\pm\frac{2\pi}{5}$. From this we easily find that

$$a_n = \frac{100}{\pi^2 n^2}\left\{-1 - \cos n\pi + 2\cos\frac{2n\pi}{5}\right\}$$

from which we obtain

$$f(t) = 17 + 6{\cdot}26 \cos t - 9{\cdot}16 \cos 2t - 1{\cdot}82 \cos 3t - {\cdot}87 \cos 4t + {\cdot}81 \cos 5t - {\cdot}39 \cos 6t - {\cdot}33 \cos 7t - {\cdot}57 \cos 8t - {\cdot}08 \cos 9t + \ldots$$

as the Fourier analysis, in contrast with

$$f(t) = 17 + 6{\cdot}31 \cos t - 9{\cdot}47 \cos 2t - 1{\cdot}96 \cos 3t - 1{\cdot}99 \cos 4t + 1{\cdot}00 \cos 5t - {\cdot}53 \cos 6t - {\cdot}51 \cos 7t - 1{\cdot}01 \cos 8t + {\cdot}16 \cos 9t$$

as the simplest function passing through the given points.

Schedule III, on page 88, for the analysis of Class III functions, is also constructed in a very similar manner. This Schedule, as far as the sine harmonics go, is exactly

like Schedule I for the odd harmonics. A separate Schedule can, if desired, be constructed for the cosine harmonics, but these two Schedules may be put together, as it will be found that the terms in one Schedule occupy the spaces left vacant in the other. For this purpose, it is necessary to have a column of the numerical cosine factors adjacent to that of sine factors. Further, it is necessary that the $v$'s should be distinguished from the $u$'s in the same column so that both sets may be added up separately. It is best if all the $v$'s, and the cosine factors, are written in red ink; or, alternatively, the $u$'s kept to the left ~~right~~ of each column and the $v$'s to the left. right Care must be taken that the $v$'s are not multiplied by the sine factors by mistake, which is not likely if different coloured inks are used.

Class IV functions present no difficulty whatever. If arranged as an odd function, they may be equally well worked on Schedule I or III and only half fill the Schedule as the "differences" vanish. If arranged as an even function they may be equally well worked on Schedules II or III, and in this case the "sums" of the ordinates vanish. Page 89 illustrates an odd Class IV function worked on Schedule III. The given ordinates lie on the curve $f(t) = \dfrac{8 \times 14^2}{\pi^2} t(\pi - t)$, from which it will easily be found that the Fourier analysis of the function is

$$f(t) = \frac{64 \times 14^2}{\pi^3}\left\{ \sin t + \frac{\sin 3t}{3^3} + \frac{\sin 5t}{5^3} + \ldots \right\}$$

or

$$\begin{aligned} f(t) = {} & 404{\cdot}6 \sin t + 15{\cdot}0 \sin 3t + 3{\cdot}24 \sin 5t \\ & + 1{\cdot}18 \sin 7t + {\cdot}56 \sin 9t + {\cdot}30 \sin 11t + {\cdot}18 \sin 13t \\ & + \ldots; \end{aligned}$$

with which the result

$$f(t) = 404{\cdot}6 \sin t + 14{\cdot}9 \sin 3t + 3{\cdot}2 \sin 5t$$
$$+ 1{\cdot}2 \sin 7t + {\cdot}6 \sin 9t + {\cdot}3 \sin 11t + {\cdot}2 \sin 13t$$

obtained by Harmonic Analysis is in excellent agreement. The agreement with the Fourier analysis is seen to be very much better than in the case of the examples worked on Schedules I and II. This is partly due to the fact that we have used 14 ordinates in place of 10; but is chiefly because, with this function, the discontinuities are in its second differential coefficient and not its first. The same function analysed by the use of 10 ordinates per half-period, but with the maximum ordinate reduced to 250, gives

$$f(t) = 258{\cdot}0 \sin t + 9{\cdot}5 \sin 3t + 2{\cdot}0 \sin 5t$$
$$+ {\cdot}7 \sin 7t + {\cdot}2 \sin 9t$$

against the Fourier analysis of

$$f(t) = 258{\cdot}0 \sin t + 9{\cdot}60 \sin 3t + 2{\cdot}07 \sin 5t$$
$$+ {\cdot}75 \sin 7t + {\cdot}35 \sin 9t + \ldots$$

There remains now only the general periodic function to be dealt with. This is easily analysed by resolving into its odd and even components. We measure ordinates over a range from $-\pi$ to $+\pi$. If $y_q$ and $y_{-q}$ are any ordinates at $t$ and $-t$ then $\dfrac{y_q + y_{-q}}{2}$ is the corresponding ordinate of the even component and $\dfrac{y_q - y_{-q}}{2}$ that of the odd component. These two functions are then analysed on Schedules II and I respectively.

We frequently, in Applied Mathematics, when we desire to calculate a curve to pass among a number of experimentally obtained points, have a much larger number of points than disposable constants in the equation of the curve selected. In this case we cannot, in general, make the curve pass *through* all the points :

instead, we determine the constants so that the sum of the *squares of the differences* between the calculated and observed values of $y$, is a minimum for all possible variations in the constants. If the curve selected be represented by a Fourier's series containing all harmonics up to the $n$th, when $n$ is insufficient to make it pass through all the points, it may be proved that the values obtained for the coefficient of any harmonic in this way is the same as if we had assumed an equation *with* sufficient harmonics as in the first part of this chapter. The proof of this will present no difficulty to the student familiar with the Method of Least Squares. Hence in dealing with periodic functions the difficulty of determining the constants of an equation (when we have less constants than points) so as to represent the observations as closely as possible does not arise: we simply adopt the ordinary method of harmonic analysis for the lower harmonics and just neglect those harmonics which are higher than we want to consider.

### *A Graphical Method of Harmonic Analysis.*

Harmonic analysis may be performed graphically by the aid of diagrams which save the necessity of constructing Schedules. The accuracy in this case, though rather less, is often sufficient for practical purposes. Fig. 19 shows the method applied to 12 ordinates per period. A large sheet of millimetre paper is taken and 6 lines inclined at 30° to one another are drawn as shown.

These lines must be drawn as accurately as possible, their inclinations being set off by the tangents or cotangents of the angles, using as large a base as the paper permits, and not by a protractor. Three concentric circles are then also drawn as shown. The intersections of the radii with the smallest circle are

then numbered 0, 1, . . . 11 consecutively. The intersections with the intermediate circle are alternately numbered 0, 1, up to 11 and left blank. Finally, every third radius is numbered on the largest circle from 0 to 11 with the result that each numbered radius in this case bears three numbers.

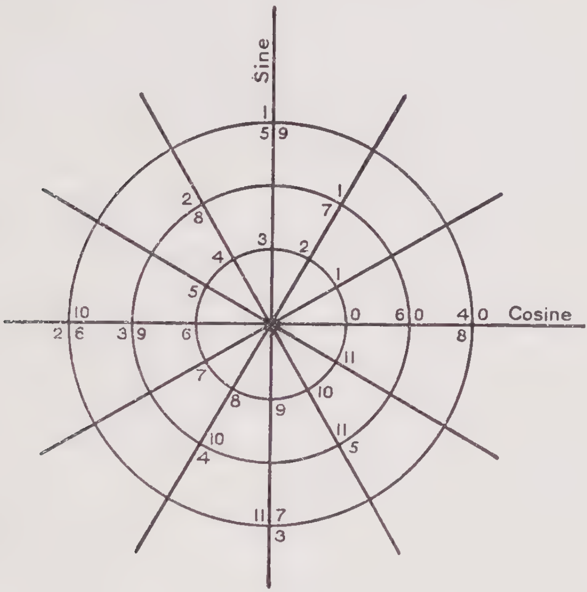


FIG. 19.

The harmonic analysis is then performed as follows: We measure the ordinate $A_0$, given either by figures or graph, along O$x$ and read its length in millimetres; then we measure off $A_1$ along radius marked 1 on the small circle and read off its projections both on O$x$ and O$y$; we likewise measure $A_2$ along the second radius to the same circle and read its projections, and so on up to $A_{11}$. The sum of all the projections along O$x$ divided by 6 gives $a_1$, while the sum of all

the projections along $Oy$ divided by 6 gives $b_1$. If instead of adding the sums of the *odd* terms to those of the *even* ones, as we have just done to get $a_1$ and $b_1$, we subtract them, we get $a_5$ and $b_5$.

If we measure $A_0$, $A_1$, $A_2$ . . . along the radii marked 0, 1, 2, . . . on the intermediate circle, and add up the projections and divide by 6 as before, we get $a_2$ and $b_2$, while if we subtract the sum of the even terms from those of the odd terms we get $a_4$ and $b_4$. Likewise $a_3$ and $b_3$ are obtained by using the radii to the largest circle, while $a_0/2$ is one-twelfth the sum of all the ordinates.

## *The Error of Harmonic Analysis.*

It is important to have some idea of the deviations of the a's and b's found by harmonic analysis from the true Fourier Constants the *a*'s and the *b*'s. We shall now show that the a's and b's can be expressed in terms of the *a*'s and *b*'s by means of the following

**Theorem.** If $2p$ ordinates per period be taken at $0, \frac{\pi}{p}, \frac{2\pi}{p}$ . . . and if each ordinate be multiplied by the value of $\cos nt$ at that ordinate, the sum of the products $\left(\text{symbolically denoted by } \sum_{q=1}^{2p-1} A_q \cos qn\pi/p\right)$ is equal to

$$p\{a_n + a_{2p-n} + a_{2p+n} + a_{4p-n} + a_{4p+n} + \dots\} \quad . \quad . \quad . \quad (15)$$

and if each ordinate be multiplied by the value of $\sin nt$ at that ordinate, the corresponding sum is

$$p\{b_n - b_{2p-n} + b_{2p+n} - b_{4p-n} + b_{4p+n} - \dots\} \quad . \quad . \quad . \quad (16)$$

For we have

$$A_q \cos q\pi n/p = \left\{ \frac{a_0}{2} + a_1 \cos q\pi/p + \dots \right.$$
$$\left. + b_1 \sin q\pi/p + \dots \right\} \cos qn\pi/p,$$

and if the trigonometrical products be replaced by the sum or difference of two terms in the usual way, we see that the coefficient of $b_r$ in the given sum is always zero, and the coefficient of $a_r$ is zero unless $r - n$ or $r + n$ is zero or a multiple of $2p$, when it is equal to $\frac{1}{2} \cdot 2p$; from which it easily follows that the required sum is equal to $p(a_n + a_{2p-n} + a_{2p+n} + \dots)$.

The second result follows in the same manner, but since the product of two sines is equal to the difference of two cosines, the coefficients of the $b$'s are alternately $+$ and $-$ giving $p(b_n - b_{2p-n} + b_{2p+n} - \dots)$.

Since the left-hand side of (15) gives $p\mathrm{a}_n$, we see at once that, in particular,

$$\left.\begin{aligned}
\frac{\mathrm{a}_0}{2} &= \frac{a_0}{2} + a_{2p} + a_{4p} + a_{6p} + \dots \\
\mathrm{a}_1 &= a_1 + a_{2p-1} + a_{2p+1} + a_{4p-1} + a_{4p+1} + \dots \\
\mathrm{a}_2 &= a_2 + a_{2p-2} + a_{2p+2} + \dots \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\mathrm{a}_{p-1} &= a_{p-1} + a_{p+1} + a_{3p-1} + a_{3p+1} + a_{5p-1} + \dots \\
\mathrm{a}_p &= a_p + a_p + a_{3p} + a_{3p} + a_{5p} + \dots \\
&\text{this last can be written} \\
\frac{\mathrm{a}_p}{2} &= a_p + a_{3p} + a_{5p} + a_{7p} + \dots
\end{aligned}\right\} \quad (17)$$

Similarly,

$$\left.\begin{aligned}
\mathrm{b}_1 &= b_1 - b_{2p-1} + b_{2p+1} - b_{4p-1} + b_{4p+1} - \dots \\
\mathrm{b}_2 &= b_2 - b_{2p-2} + b_{2p+2} - \dots \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\mathrm{b}_{p-1} &= b_{p-1} - b_{p+1} + b_{3p-1} - b_{3p+1} + \dots
\end{aligned}\right\} \quad (18)$$

If the $a$'s and the $b$'s diminish regularly and progressively, we see from these equations, that the accuracy of the a's and b's regularly diminish from $\mathrm{a}_1$ and $\mathrm{b}_1$ till $\mathrm{a}_{p-1}$ and $\mathrm{b}_{p-1}$; which latter two cannot be very accurate, since we have, roughly,

$$\mathrm{a}_{p-1} = a_{p-1} + a_{p+1} \text{ and } \mathrm{b}_{p-1} = b_{p-1} - b_{p+1}.$$

The remarkable accuracy with which $\frac{\mathrm{a}_p}{2}$ represents $a_p$ should be noted.

It should also be noted that in the equations (17) each $a$ occurs once, and only once, and similarly with the $b$'s in (18). Hence each harmonic in the function analysed affects *one*, and *one* only, of the coefficients of harmonic analysis. For instance, if we are using 10 ordinates per half-period, and if there is a large thirteenth harmonic while all the other harmonics beyond the ninth are small; all the a's and b's will be nearly correct except $\mathrm{a}_7$ and $\mathrm{b}_7$; for we have in this case $\mathrm{a}_7 = a_7 + a_{13} + a_{27} + a_{33} + a_{47} + \ldots$
and $\mathrm{b}_7 = b_7 - b_{13} + b_{27} - b_{33} + b_{47} - \ldots$;
or, approximately, $\mathrm{a}_7 = a_7 + a_{13}$ and $\mathrm{b}_7 = b_7 - b_{13}$.

*Determination of the Amplitude of any Selected Harmonic.*

We sometimes desire to find the amplitude of some particular harmonic, say the $n$th, which we are interested in, without troubling at all about any other harmonic. If the harmonic is completely unknown, three ordinates per period of the harmonic is the least which will suffice to determine it, although if we only want either the sine or the cosine component two ordinates per period will suffice. With a larger number of ordinates per period, the harmonic can naturally be determined more exactly. We shall give formulæ for the cases of 3, 4, or 6 ordinates in the period of the harmonic respectively, which will at the same time indicate the error of determining a harmonic in this manner.

If in the formula

$$\sum_{q=0}^{2p-1} A_q \cos q\frac{n\pi}{p} = p\{a_n + a_{2p-n} + a_{2p+n} + \ldots\} \quad (19)$$

we put $2p = 3n$, so that the ordinates are at intervals of $2\pi/3n$, we get

$$A_0 - \frac{A_1}{2} - \frac{A_2}{2} + A_3 - \frac{A_4}{2} - \frac{A_5}{2} + A_6 - \ldots$$
$$= \frac{3n}{2}\left\{a_n + a_{2n} + a_{4n} + a_{5n} + a_{7n} + \ldots\right\};$$

and if in the formula

$$\sum_{q=0}^{2p-1} A_q \sin q\frac{n\pi}{p} = p\{b_n - b_{2p-n} + b_{2p+n} - \ldots\} \quad (20)$$

we make the same substitution, we have

$$\frac{\sqrt{3}}{2}\left\{A_1 - A_2 + A_4 - A_5 + A_7 - A_8 + \ldots\right\}$$
$$= \frac{3n}{2}\left\{b_n - b_{2n} + b_{4n} - b_{5n} + b_{7n} - \ldots\right\}$$

If now the $2n$th, $4n$th, $5n$th . . . harmonics are negligible compared with the $n$th, these formulæ give us approximately $a_n$ and $b_n$, using three ordinates per period of harmonic.

If in the formulæ (19) and (20) we put $2p = 4n$, that is, if we use four ordinates in the period of the harmonic, we get

$$A_0 - A_2 + A_4 - A_6 + \ldots = 2n\{a_n + a_{3n} + a_{5n} + a_{7n} + \ldots\}$$

and

$$A_1 - A_3 + A_5 - A_7 + \ldots = 2n\{b_n - b_{3n} + b_{5n} - b_{7n} + \ldots\}$$

which give $a_n$ and $b_n$ provided the $3n$th and higher harmonics are negligible.

If we put $2p = 6n$, that is, use six ordinates per period of the $n$th harmonic, we get similarly

$$A_0 + \frac{A_1}{2} - \frac{A_2}{2} - A_3 - \frac{A_4}{2} + \frac{A_5}{2} + A_6 + \ldots$$
$$= 3n\{a_n + a_{5n} + a_{7n} + a_{11n} + a_{13n} + \ldots\}$$

$$\text{and } \frac{\sqrt{3}}{2}\left\{A_1 + A_2 - A_4 - A_5 + A_7 + A_8 - \ . \ . \ .\right\}$$
$$= 3n\{b_n - b_{5n} + b_{7n} - b_{11n} + b_{13n} - \ . \ . \ .\}$$

which formulæ give us $a_n$ and $b_n$ provided the $5n$th and higher harmonics are negligible. Similar formulæ for eight or any other number of ordinates per period can easily be written down. The simplification of these formulæ for the four special classes of periodic functions may be left to the student.

As an example, suppose we wish to find the amplitude of the third harmonic in the odd Class IV function defined by $f(t) = \frac{18t}{\pi}$ from $t = 0$ to $t = \frac{\pi}{2}$. If we use four ordinates per period of the third harmonic, that is twelve per period of the function, the ordinates over the first quarter of a period are 0, 3, 6, 9, so we easily find

$$-6 = 6\{b_3 - b_9 + b_{15} \ . \ . \ .\} \quad \text{or } b_3 = -1 \text{ approx.}$$

Using six ordinates per period of the harmonic we find that the ordinates over half a period of the function are 0, 2, 4, 6, 8, 8, 6, 4, 2, 0, from which we find

$$\frac{\sqrt{3}}{2}(-8) = 9\left\{b_3 - b_{15} + b_{21} - \ . \ . \ .\right\}$$

or
$$b_3 = -\frac{4}{9}\sqrt{3} = -\cdot 770.$$

Actually, $b_3$ for this function is $-\frac{8}{\pi^2} = -\cdot 811$.

### *A Simple Method of Harmonic Analysis.*

There is also a modification of this method of harmonic analysis, which we do not consider very satisfactory, but which may be employed when we are

given a graph of the function to be analysed. In this modification we draw different sets of equidistant ordinates, and use one set of ordinates for determining one coefficient and another set for determining another, choosing the ordinates in each case so that we have a definite (small) number of ordinates to the period of each harmonic. The work involved in placing and carefully measuring such a large number of ordinates renders, we consider, this method more laborious than the one described, while the results are generally less accurate. The only saving is in the simplification of the sine and cosine factors—or their abolition in the case when only two ordinates per harmonic are chosen and placed at the positions of the maximum and minimum values of such harmonic.

This method may conveniently be developed from the formulæ connecting the ordinates with the coefficients. If, for instance, we have a general function and neglect all harmonics beyond the third and take 12 ordinates per period it can easily be seen that we have the following equations for the coefficients:

$$\frac{a_0}{2} = \tfrac{1}{12}(A_0 + A_1 + A_2 + \ldots + A_{10} + A_{11})$$
$$a_3 = \tfrac{1}{6}(A_0 - A_2 + A_4 - A_6 + A_8 - A_{10})$$
$$b_3 = \tfrac{1}{6}(A_1 - A_3 + A_5 - A_7 + A_9 - A_{11})$$
$$a_2 = \tfrac{1}{4}(A_0 - A_3 + A_6 - A_9)$$
$$b_2 = \tfrac{1}{4}(A_{1\cdot 5} - A_{4\cdot 5} + A_{7\cdot 5} - A_{10\cdot 5})$$
$$a_1 + a_3 = \tfrac{1}{2}(A_0 - A_6)$$
$$b_1 - b_3 = \tfrac{1}{2}(A_3 - A_9)$$

Here $A_{1\cdot 5}$ means the ordinate half-way between $A_0$ and $A_3$ and similarly for the other terms in the same line. To use these formulæ, we thus have to place and measure sixteen different ordinates, while the same number of equally spaced ordinates would, with the ordinary method of harmonic analysis, enable us to

determine the first seven harmonics. All that has been saved is the trouble of multiplying the ordinates by the sine factors—a matter of very little trouble with a slide rule.

For the general periodic function, a sufficiently accurate result for rough work can sometimes be obtained by assuming that all the harmonics beyond the third are negligible. Eight ordinates are then sufficient and it will be found that the ordinary formulæ for harmonic analysis give:

$$
\begin{aligned}
8 \cdot \frac{a_0}{2} &= A_0 + A_2 + A_4 + A_6 + A_1 + A_3 + A_5 + A_7 \\
4a_1 &= A_0 - A_4 + \cdot 707(A_1 + A_7 - A_3 - A_5) \\
4a_2 &= A_0 + A_4 - A_2 - A_6 \\
4a_3 &= A_0 - A_4 - \cdot 707(A_1 + A_7 - A_3 - A_5) \\
8 \cdot \frac{a_4}{2} &= (A_0 + A_2 + A_4 + A_6) - (A_1 + A_3 + A_5 + A_7) \\
4b_1 &= A_2 - A_6 + \cdot 707(A_1 + A_3 - A_5 - A_7) \\
4b_2 &= A_1 + A_5 - A_3 - A_7 \\
4b_3 &= -(A_2 - A_6) + \cdot 707(A_1 + A_3 - A_5 - A_7)
\end{aligned}
$$

### *A New Method of Harmonic Analysis.*

We will now explain a totally different method of harmonic analysis which, though it is rather more laborious than the ordinary method, possesses considerable advantage from the point of view of accuracy and is quite easy to apply when it has become familiar. As mentioned above, the method consists in joining the tops of the given ordinates with straight lines or by parabolas of different degrees. This can be done in quite a variety of manners, but the three varieties of the method given probably include the most useful that can be obtained.

Let us, as usual, take $2p$ equally spaced ordinates $y_0, y_1, \ldots$ per period at $0, \pi/p, \ldots$ and for brevity

let $\pi/p$ be again denoted by $\alpha$. Let us join the tops of the ordinates by straight lines; then the slope between $y_0$ and $y_1$ is $\dfrac{y_1 - y_0}{\alpha}$ and that between $y_2$ and $y_1$ is $\dfrac{y_2 - y_1}{\alpha}$; so that there is a discontinuity of slope at $y_1$ of $\dfrac{y_2 - 2y_1 + y_0}{\alpha}$, that is, of $\dfrac{p\Delta^2 y_1}{\pi}$ where $\Delta^2 y_1$ is that second difference of the $y$'s which is vertically below $y_1$. Substituting this value for $\Gamma'_\alpha$ in the formulæ (15) and (16) on p. 59* we get

$$\left.\begin{aligned} a_n &= -\frac{p}{\pi^2 n^2}\Sigma\Delta^2 y_1 \cos\frac{n\pi}{p} \\ \text{and} \qquad b_n &= -\frac{p}{\pi^2 n^2}\Sigma\Delta^2 y_1 \sin\frac{n\pi}{p} \end{aligned}\right\} \quad . \quad . \quad . \quad (21)$$

But $\dfrac{1}{p}\Sigma\Delta^2 y_1 \cos\dfrac{n\pi}{p}$ is the $\mathrm{a}_n$ of ordinary harmonic analysis *for the function whose ordinates are* $\Delta^2 y_0$, $\Delta^2 y_1$, . . . .

Hence we have

$$\left.\begin{aligned} a_n &= -\frac{p^2}{\pi^2 n^2}\mathrm{a}_n \\ \text{and} \qquad b_n &= -\frac{p^2}{\pi^2 n^2}\mathrm{b}_n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (22)$$

The quantities $\mathrm{a}_n$ and $\mathrm{b}_n$ can be calculated in the usual manner on one of the given Schedules so that, in labour, this method of harmonic analysis exceeds the ordinary method only by that involved in forming the second differences of the ordinates. The Schedules,

* The student must not confuse the present use of $\alpha$ with its use in these formulæ. In (15) and (16) $\alpha$ was used to specify the position of one of the discontinuities; now it is being used to indicate the width between successive ordinates; but no confusion need occur through this double use of this very convenient letter.

of course, only give $a_n$ and $b_n$ when $n \ngtr p$, but if we write $p + n$ in the expression for $a_n$ we get

$$a_{p+n} = \frac{1}{p}\Sigma\Delta^2 y_q \cos\frac{q(p+n)\pi}{p}$$

which is the same as if we had written $p - n$, hence we have $a_{p+n} = a_{p-n}$ and similarly $b_{p+n} = - b_{p-n}$, and we see also that $a_{2p+n} = a_n$ and $b_{2p+n} = b_n$, hence we are able to write down the complete analysis of the function defined by the straight lines joining the tops of the ordinates. In actual practice, of course, the function to be analysed does not consist of these straight lines and so the harmonics beyond the $p$th are too inaccurate to trouble about.

As a very simple example, let us take the function analysed in Schedule I, page 85. If we form the second differences of the ordinates we find them all zero save at $\frac{2\pi}{5}$, where we have $- 10$, and at $-\frac{2\pi}{5}$, where we have $+ 10$.

Hence we get

$$b_n = -\frac{10}{\pi^2 n^2}\left\{-10\sin\frac{2\pi n}{5} + 10\sin\left(-\frac{2\pi n}{5}\right)\right\},$$

or

$$b_n = \frac{200}{\pi^2 n^2}\sin\frac{2n\pi}{5},$$

agreeing exactly with the harmonic analysis given on page 91, which it obviously should do.

In a second manner of applying this method we draw a parabolic arc through the tops of the three ordinates $y_0, y_1, y_2$; another one through the tops of $y_2, y_3, y_4$ and so on; we thus get a function with discontinuities at the *even* ordinates $y_0, y_2, y_4 \ldots$, and at these ordinates there will, in general, be a discontinuity both of slope and of curvature. We must find an expression for the magnitude of these discontinuities.

If we take the origin at $y_1$ it will easily be found that the equation of the parabola through the tops of $y_0$, $y_1$ and $y_2$ is

$$y = y_1 + \frac{y_2 - y_0}{2a} . t + \frac{y_2 - 2y_1 + y_0}{2a^2} . t^2,$$

where $a$ is, as usual, the distance between the ordinates. Similarly, the equation of the parabola joining $y_2$, $y_3$, and $y_4$ referred to an origin at $y_3$ is

$$y = y_3 + \frac{y_4 - y_2}{2a} . t + \frac{y_4 - 2y_3 + y_2}{2a^2} . t^2.$$

Differentiating and putting $t = - a$, we see that this parabola has a slope of

$$\frac{- y_4 + 4y_3 - 3y_2}{2a}$$

at $y_2$.

Similarly, it will be found that the first parabola has a slope of

$$\frac{3y_2 - 4y_1 + y_0}{2a}$$

at the same ordinate; and so the discontinuity of slope at $y_2$ is

$$\frac{- y_4 + 4y_3 - 6y_2 + 4y_1 - y_0}{2a},$$

or $-\dfrac{\Delta^4 y_2}{2a}$, where $\Delta^4 y_2$ is the fourth difference of the $y$'s which occurs vertically below $y_2$.

We must now find the discontinuity in $\dfrac{d^2y}{dt^2}$. For the second parabola we see that $\dfrac{d^2y}{dt^2} = \dfrac{y_4 - 2y_3 + y_2}{\alpha^2}$, or in our difference notation, $\dfrac{\Delta^2 y_3}{\alpha^2}$. Similarly, $\dfrac{d^2y}{dt^2}$

for the first parabola is $\frac{\Delta^2 y_1}{\alpha^2}$; and hence the discontinuity in $\frac{d^2y}{dt^2}$ at the ordinate $y_2$ is $\frac{\Delta^2 y_3 - \Delta^2 y_1}{\alpha^2}$, that is, the difference between the second differences which lie each side of that one below $y_2$. Replacing $\alpha$ by $\pi/p$ and substituting in the formulæ (15) and (16) of page 59, we get

$$a_n = \frac{p}{2\pi^2 n^2}\Sigma\Delta^4 y_2 \cos\frac{2n\pi}{p} + \frac{p^2}{\pi^3 n^3}\Sigma(\Delta^2 y_3 - \Delta^2 y_1)\sin\frac{2n\pi}{p} \quad . \quad . \quad . \quad (23)$$

and

$$b_n = \frac{p}{2\pi^2 n^2}\Sigma\Delta^4 y_2 \sin\frac{2n\pi}{p} - \frac{p^2}{\pi^3 n^3}\Sigma(\Delta^2 y_3 - \Delta^2 y_1)\cos\frac{2n\pi}{p} \quad . \quad . \quad . \quad (24)$$

The summation extends to all the ordinates *at which there are discontinuities.* The expressions actually written under the $\Sigma$'s represent the discontinuities at $y_2$ only.

If $a'_n$ and $b'_n$ are the ordinary harmonic analysis coefficients of the function with ordinates $\Delta^4 y_2$ at $y_2$, etc., and $a''_n$ and $b''_n$ are those of the function with ordinates $\Delta^2 y_3 - \Delta^2 y_1$ * at $y_2$, etc., we may write these coefficients

$$a_n = \frac{p^2}{2\pi^2 n^2}a'_n + \frac{p^3}{\pi^3 n^3}b''_n \quad . \quad . \quad . \quad (25)$$

and

$$b_n = \frac{p^2}{2\pi^2 n^2}b'_n - \frac{p^3}{\pi^3 n^3}a''_n \quad . \quad . \quad . \quad (26)$$

* With the obvious notation of using the suffix 1·5 to indicate a term standing mid-way between terms of suffixes 1 and 2 we have $\Delta^2 y_3 - \Delta^2 y_1 = \Delta^3 y_{2\cdot5} + \Delta^3 y_{1\cdot5}$ while $\Delta^4 y_2 = \Delta^3 y_{2\cdot5} - \Delta^3 y_{1\cdot5}$.

As an example of this, we will take the case of an even Class IV function whose ordinates are 2·000, 1·732, 1·000 and 0 at 0, $\frac{\pi}{6}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$ respectively. Owing to the symmetry, we need only obtain the differences over a quarter of a period, but to do so we must write down two ordinates before and after the quarter period chosen. We thus get the Table

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t$ | $-\frac{\pi}{3}$ | | $-\frac{\pi}{6}$ | | 0 | | $\frac{\pi}{6}$ | | $\frac{\pi}{3}$ | | $\frac{\pi}{2}$ | | $\frac{2\pi}{3}$ | | $\frac{5\pi}{6}$ |
| $y$ | 1·000 | | 1·732 | | 2·000 | | 1·732 | | 1·000 | | 0 | | −1·000 | | −1·732 |
| $\Delta y$ | | ·732 | | ·268 | | −·268 | | −·732 | | −1·000 | | −1·000 | | −·732 | |
| $\Delta^2 y$ | | | −·464 | | −·536 | | −·464 | | −·268 | | 0 | | ·268 | | |
| $\Delta^3 y$ | | | | −·072 | | ·072 | | ·196 | | ·268 | | ·268 | | | |
| $\Delta^4 y$ | | | | | ·144 | | ·124 | | ·072 | | 0 | | | | |
| $\Delta^2 y_3 - \Delta^2 y_1$ | | | | | 0 | | ·268 | | ·464 | | ·536 | | | | |

Thus, in a quarter of a period, we have two discontinuities in $\Delta^4 y$, viz., at 0 and $\frac{\pi}{3}$. In a complete period, there are two discontinuities similar to the first, and four similar to the second. In $\Delta^2 y_3 - \Delta^2 y_1$ we only have one discontinuity in the quarter period, viz., at $\frac{\pi}{3}$. Substituting in (23) and remembering that $n$ is odd and $p = 6$, we get

$$a_n = \frac{6}{2\pi^2 n^2}\left\{2 \times \cdot 144 \cos 0 + 4 \times \cdot 072 \cos \frac{n\pi}{3}\right\} + \frac{36}{\pi^3 n^3}\left\{4 \times \cdot 464 \sin \frac{n\pi}{3}\right\}.$$

Working out this formula, we find

$$\begin{aligned} f(t) = 1{\cdot}9975 \cos t &- \cdot 0097 \cos 5t + \cdot 0081 \cos 7t \\ &- \cdot 00033 \cos 11t + \cdot 00163 \cos 13t \\ &+ \cdot 00007 \cos 17t + \cdot 00064 \cos 19t + \ldots \end{aligned}$$

which agrees remarkably closely with the simple curve $f(t) = 2 \cos t$ which the given points evidently lie on.

As an improvement in this method we may also

draw parabolic arcs joining the ordinates $y_1$, $y_2$, $y_3$ and $y_3$, $y_4$, $y_5$, etc., and then add the analysis of this case to that of the former case and divide by two. This will be allowed for if we sum the expressions in (23) and (24) for *every* ordinate, instead of only for the even ordinates, and divide the result by two. We leave it as an exercise for the student to apply this method to the above example and find that the agreement with $2 \cos t$ is made considerably better.

A third manner of applying this method of harmonic analysis is to draw a curve joining the tops of ordinates $y_1$ and $y_2$ such that: at $y_1$ the curve is parallel to the straight line joining the tops of $y_0$ and $y_2$, and at $y_2$ it is parallel to the straight line joining the tops of $y_1$ and $y_3$. Since this is virtually drawing a curve through four points, the curve must be a parabola of the third degree. The tops of all the other consecutive ordinates are joined in a like manner. This function then has no discontinuities either in magnitude or slope; but it has discontinuities in its second and third differential coefficients.

We leave it to the student to prove that the equation of the curve joining $y_2$ and $y_3$ is

$$y = y_2 + (y_3 - y_1)\frac{t}{2a} + (-y_4 + 4y_3 - 5y_2 + 2y_1)\frac{t^2}{2a^2} + (y_4 - 3y_3 + 3y_2 - y_1)\frac{t^3}{2a^3}$$

the origin being taken on $y_2$. The equation of the arc joining $y_1$ and $y_2$ can be written down from this by writing $y_1$ for $y_3$, $y_0$ for $y_4$ and changing $t$ into $-t$; and so is

$$y = y_2 + (y_3 - y_1)\frac{t}{2a} + (-y_0 + 4y_1 - 5y_2 + 2y_3)\frac{t^2}{2a^2} - (y_0 - 3y_1 + 3y_2 - y_3)\frac{t^3}{2a^3}$$

referred to the same origin.

The third differential coefficient of the first curve is constant and $= 3(y_4 - 3y_3 + 3y_2 - y_1)/a^3$, while for the second it is $3(y_3 - 3y_2 + 3y_1 - y_0)/a^3$. Hence the discontinuity in this coefficient at $y_2$ is $3(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)/a^3$; that is, $3\Delta^4 y_2/a^3$ in our previous difference notation.

The second differential coefficient of the first curve at $t = 0$ is $(- y_4 + 4y_3 - 5y_2 + 2y_1)/a^2$, while for the second curve at the same point it is $(2y_3 - 5y_2 + 4y_1 - y_0)/a^2$. Hence the discontinuity in this coefficient at $y_2$ is $(- y_4 + 2y_3 - 2y_1 + y_0)/a^2$; that is, it is $(- \Delta^2 y_3 + \Delta^2 y_1)/a^2$. If we reduce these suffixes by unity so as to get the discontinuities at the ordinate $y_1$ and then substitute in the formulæ (15) and (16) of page 59 we get

$$a_n = - \frac{p^2}{\pi^3 n^3}\Sigma(\Delta^2 y_2 - \Delta^2 y_0) \sin \frac{n\pi}{p} + \frac{3p^3}{\pi^4 n^4}\Sigma\, \Delta^4 y_1 \cos \frac{n\pi}{p} \quad . \quad . \quad . \quad (27)$$

and

$$b_n = \frac{p^2}{\pi^3 n^3}\Sigma(\Delta^2 y_2 - \Delta^2 y_0) \cos \frac{n\pi}{p} + \frac{3p^3}{\pi^4 n^4}\Sigma\, \Delta^4 y_1 \sin \frac{n\pi}{p} \quad . \quad . \quad . \quad (28)$$

Or if $a'_n$ and $b'_n$ are the ordinary coefficients of harmonic analysis for the function with ordinates $\Delta^4 y_1$ at $y_1$, etc., and $a''_n$ and $b''_n$ similarly for the function with ordinates $\Delta^2 y_2 - \Delta^2 y_0$ at $y_1$, etc., we have

$$a_n = - \frac{p^3}{\pi^3 n^3} b''_n + \frac{3p^4}{\pi^4 n^4} a'_n . \quad . \quad . \quad (29)$$

and

$$b_n = \frac{p^3}{\pi^3 n^3} a''_n + \frac{3p^4}{\pi^4 n^4} b'_n \quad . \quad . \quad . \quad (30)$$

It is noteworthy that we are concerned here with *exactly the same differences* as we were when dealing with ordinary parabolic arcs joining three ordinates.

Applying this case to the same example as we took previously (page 107), we get

$$a_n = -\frac{36}{\pi^3 n^3}\left\{4 \times \cdot 268 \sin\frac{n\pi}{6} + 4 \times \cdot 464 \sin\frac{n\pi}{3} + 2 \times \cdot 536 \sin\frac{n\pi}{2}\right\}$$

$$+ \frac{648}{\pi^4 n^4}\left\{2 \times \cdot 144 \cos 0 + 4 \times \cdot 124 \cos\frac{n\pi}{6} + 4 \times \cdot 072 \cos\frac{n\pi}{3}\right\}$$

Working this out, which is not the trouble it appears to be, for after having worked out $a_1$ there is very little to do save divide the same figures by $n^3$ and $n^4$, we get

$$\begin{aligned} f(t) = {} & 1\cdot 99814 \cos t + \cdot 00002 \cos 5t + \cdot 00138 \cos 7t \\ & + \cdot 000001 \cos 11t + \cdot 000031 \cos 13t \\ & + \cdot 000000 \cos 17t + \ldots \qquad (31) \end{aligned}$$

a remarkably good approximation to $2 \cos t$. It may be objected that the ordinary method of harmonic analysis would have given an even better approximation as it would have given $2 \cos t$ exactly. Quite so, but that is accidental and due to the fact that the selected points lie exactly on the curve $2 \cos t$ which is one of the terms of the assumed harmonic expansion. Had we, on the other hand, used the very similar function $8t(\pi - t)/\pi^2$ and taken 12 ordinates per period and joined successive even ordinates by a parabolic arc passing through the intermediate odd ones, our present method would, in this case, have given exact agreement with the Fourier analysis of the function, which is

$$\begin{aligned} f(t) = {} & 2\cdot 0641 \sin t + \cdot 07627 \sin 3t + \cdot 01651 \sin 5t \\ & + \cdot 00602 \sin 7t + \cdot 00283 \sin 9t + \ldots; \end{aligned}$$

whereas the ordinary method of harmonic analysis with the same number of ordinates would have given us

$$f(t) = 2{\cdot}0577 \sin t + {\cdot}07407 \sin 3t + {\cdot}01630 \sin 5t,$$

the higher harmonics being quite undeterminable. The accuracy of this is "miles behind" the accuracy with which (31) represents $2 \cos t$.

In the last manner of applying the present method the labour is about twice that of ordinary harmonic analysis apart from the labour of finding the fourth differences, since we have two different discontinuities at each ordinate and each discontinuity has to be treated just as if it were an ordinate. The present method is not put forward for use where any rough analysis is good enough, but only when the highest accuracy is required, and especially in those cases where we require the amplitudes of one or more of the lower harmonics very accurately, together with as high an accuracy as is obtainable, on the higher harmonics. The last variety of the method described should then always be employed.

The student who ever has any practical harmonic analysis to do will soon find he has had ample exercise in the subject. We would recommend all students, however, to construct one or two Schedules for a different number of ordinates per half-wave-length than those we have given.

### *Interpolation Formulæ for Periodic Functions.*

When the value of a periodic function is known for non-equidistant ordinates the harmonic analysis can be performed by first writing down an interpolation formula and then transforming it by the method described in Chapter IV.

If we have $n$ points, $(a,A)$, $(b,B)$ . . . $(k,K)$ $(l,L)$

the equation of the parabola of the $n(-1)$th degree passing through the points is evidently

$$f(x) = \frac{(x-b)(x-c)\ldots(x-l)}{(a-b)(a-c)\ldots(a-l)}\,A + \ldots$$
$$+ \frac{(x-a)(x-b)\ldots(x-k)}{(l-a)(l-b)\ldots(l-k)}\,L \ldots \qquad (32)$$

If now we imagine all these points lying between 0 and $2\pi$ and imagine the whole set repeated an infinite number of times at intervals of $2\pi$ to both $-\infty$ and $+\infty$, we find that in the limit (32) becomes

$$f(x) = \frac{\sin\frac{1}{2}(x-b)\ldots\sin\frac{1}{2}(x-l)}{\sin\frac{1}{2}(a-b)\ldots\sin\frac{1}{2}(a-l)}\,A + \ldots$$
$$+ \frac{\sin\frac{1}{2}(x-a)\ldots\sin\frac{1}{2}(x-k)}{\sin\frac{1}{2}(l-a)\ldots\sin\frac{1}{2}(l-k)}\,L \ldots \qquad (33)$$

when $n$ is odd and

$$f(x) = \cos\tfrac{1}{2}(x-a)\,\frac{\sin\frac{1}{2}(x-b)\ldots\sin\frac{1}{2}(x-l)}{\sin\frac{1}{2}(a-b)\ldots\sin\frac{1}{2}(a-l)}\,A + \ldots$$
$$+ \cos\tfrac{1}{2}(x-l)\,\frac{\sin\frac{1}{2}(x-a)\ldots\sin\frac{1}{2}(x-k)}{\sin\frac{1}{2}(l-a)\ldots\sin\frac{1}{2}(l-k)}\,L$$
$$\ldots \qquad (34)$$

when $n$ is even.

The proof of these formulæ, though rather tedious to write out, is quite easy if the following results be borne in mind:

$$\sin\frac{1}{2}x = \frac{1}{2}x\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right)\ldots$$

$$\tfrac{1}{2}\cot\frac{1}{2}x = \frac{1}{x} + \frac{1}{x+2\pi} + \frac{1}{x-2\pi} + \frac{1}{x+4\pi} + \ldots$$

$$\tfrac{1}{2}\operatorname{cosec}\frac{1}{2}x = \frac{1}{x} - \frac{1}{x+2\pi} - \frac{1}{x-2\pi} + \frac{1}{x+4\pi} + \ldots$$

The expressions in (33) and (34) obviously satisfy the conditions of passing through the given points

and, since each of the factors in the numerators merely changes sign when $x$ is increased by $2\pi$ and there is in both cases an even number of such factors, the period is evidently $2\pi$ in each case.

We know that if in (32) each given ordinate be increased by the same amount, $f(x)$ will be increased by this amount for all values of $x$; that is, the formula (32) still represents the same curve: we conclude therefore that formulæ (33) and (34) likewise satisfy this condition.

If in (33) and (34) we write $e^{\frac{ix}{2}} = \sqrt{u}$ and arrange the result in powers of $u$ it will be seen that no fractional powers occur and that further, if the result is expressed in sines and cosines of multiples of $x$, the highest harmonic when $n$ is odd is the $\frac{n-1}{2}$th and when $n$ is even is the $\frac{n}{2}$th. But these are just the harmonics which we found it necessary to assume present for our harmonic analysis when taking equidistant ordinates: we infer, therefore, that for equidistant ordinates (33) and (34) give the same result as is attained by ordinary harmonic analysis.

When the ordinates are equidistant the expressions (33) and (34) may be greatly simplified. Let the $n$ points be $(a, Y_1)$, $(2a, Y_2)$ . . . $(2\pi, Y_n)$, where $a \equiv 2\pi/n$. Then it may be shown, by a well-known factorisation result in trigonometry that (33) becomes

$$f(x) = \frac{1}{n} \sin \frac{nx}{2} \Big\{ - Y_1 \operatorname{cosec} \tfrac{1}{2}(x - a) +$$

$$Y_2 \operatorname{cosec} \tfrac{1}{2}(x - 2a) - \ldots - Y_n \operatorname{cosec} \tfrac{1}{2}(x - 2\pi) \Big\} \quad (35)$$

which holds only when $n$ is odd; while similarly (34) becomes

$$f(x) = \frac{1}{n} \sin \frac{nx}{2} \left\{ - Y_1 \cot \tfrac{1}{2}(x - a) + Y_2 \cot \tfrac{1}{2}(x - 2a) - \ldots + Y_n \cot \tfrac{1}{2}(x - 2\pi) \right\} \cdot \quad (35a)$$

which holds only when $n$ is even. That these expressions satisfy the conditions of passing through the given points and of having a period of $2\pi$ can be very easily verified. These formulæ, then, are mathematically the same as the results arrived at by ordinary harmonic analysis: they suffer from the results not being in so useful a form from a physical point of view, but they possess the advantage that they can be immediately written down without any calculation.

Formulæ (35) may be usefully employed for interpolation with non-periodic functions when these are given for a fairly large number of equidistant ordinates. They are much more easily written down and calculated from than (32) and give practically the same except near or beyond the end-points.

We will now simplify (34) in the cases of functions belonging to one of our four Special Classes.

First let $f(x)$ be an odd periodic function and let the $n$ given points $(a,A) \ldots (l,L)$ be considered as lying between 0 and $\pi$. If we make (34) in addition to passing through these points also pass through the $n$ points $(-a, -A)$, etc., it will be found to become

$$f(x) = \frac{\sin x}{\sin a} \cdot \frac{(\cos x - \cos b) \ldots (\cos x - \cos l)}{(\cos a - \cos b) \ldots (\cos a - \cos l)} A + \ldots + \frac{\sin x}{\sin l} \cdot \frac{(\cos x - \cos a) \ldots (\cos x - \cos k)}{(\cos l - \cos a) \ldots (\cos l - \cos k)} L \quad \ldots \quad (36)$$

Similarly, if $f(x)$ is an even function and if we make (34) in addition to passing through the $n$ given points also pass through the $n$ points $(-a,A)$, etc., it will become

$$f(x) = \frac{(\cos x - \cos b) \ldots (\cos x - \cos l)}{(\cos a - \cos b) \ldots (\cos a - \cos l)} \mathrm{A} + \ldots$$
$$+ \frac{(\cos x - \cos a) \ldots (\cos x - \cos k)}{(\cos l - \cos a) \ldots (\cos l - \cos k)} \mathrm{L}. \quad \ldots \quad (37)$$

Again, if $f(x)$ is an odd harmonic function, and if we make it pass in addition through the $n$ points $(a - \pi, - \mathrm{A})$, etc., it will become

$$f(x) = \frac{\sin (x - b) \ldots \sin (x - l)}{\sin (a - b) \ldots \sin (a - l)} \mathrm{A} + \ldots$$
$$+ \frac{\sin (x - a) \ldots \sin (x - k)}{\sin (l - a) \ldots \sin (l - k)} \mathrm{L}. \quad \ldots \quad (38)$$

if $n$ is even and

$$f(x) = \cos (x - a) \frac{\sin (x - b) \ldots \sin (x - l)}{\sin (a - b) \ldots \sin (a - l)} \mathrm{A} + \ldots \quad \ldots \quad (39)$$

if $n$ is odd.

Finally, if the $f(x)$ is a Class IV function and if the $n$ given points be considered as lying between 0 and $\pi/2$ and if we make (34) pass through similar points in the other three quadrants we get

$$f(x) = \frac{\sin x (\cos 2x - \cos 2b) \ldots (\cos 2x - \cos 2l)}{\sin a (\cos 2a - \cos 2b) \ldots (\cos 2a - \cos 2l)} \mathrm{A} + \ldots \quad \ldots \quad (40)$$

for an odd Class IV function and

$$f(x) = \frac{\cos x (\cos 2x - \cos 2b) \ldots (\cos 2x - \cos 2l)}{\cos a (\cos 2a - \cos 2b) \ldots (\cos 2a - \cos 2l)} \mathrm{A} + \ldots \quad \ldots \quad (41)$$

for an even Class IV function.

These results are most readily obtained from (36) and (37) respectively. All the formulæ (36) to (41) evidently satisfy all the conditions imposed upon them.

The formulæ (36) to (39) can be further simplified to half the number of factors in each term if the ordinates are chosen symmetrically so that $l = \pi - a$; $k = \pi - b$, etc. In this case, we will suppose the ordinates at $a$ and $\pi - a$ to be $A_1$ and $A_2$ respectively and similarly for all the other ordinates.

We then find that for an odd function (36) becomes

$$f(x) = \left\{\frac{A_1 + A_2}{2} \frac{\sin x}{\sin a} + \frac{A_1 - A_2}{2} \frac{\sin 2x}{\sin 2a}\right\} \times$$
$$\frac{(\cos 2x - \cos 2b) \ldots (\cos 2x - \cos 2l)}{(\cos 2a - \cos 2b) \ldots (\cos 2a - \cos 2l)} + \ldots \quad (42)$$

Likewise for an even function (37) becomes

$$f(x) = \left\{\frac{A_1 - A_2}{2} \frac{\cos x}{\cos a} + \frac{A_1 + A_2}{2} \frac{\cos 2x}{\cos 2a}\right\} \times$$
$$\frac{(\cos 2x - \cos 2b) \ldots (\cos 2x - \cos 2l)}{(\cos 2a - \cos 2b) \ldots (\cos 2a - \cos 2l)} + \ldots \quad (43)$$

While for an odd harmonic function both (38) and (39) give

$$f(x) = \left\{\frac{A_1 + A_2}{2} \frac{\sin x}{\sin a} + \frac{A_1 - A_2}{2} \frac{\cos x}{\cos a}\right\} \times$$
$$\frac{(\cos 2x - \cos 2b) \ldots (\cos 2x - \cos 2l)}{(\cos 2a - \cos 2b) \ldots (\cos 2a - \cos 2l)} + \ldots \quad (44).$$

In these three formulæ we have supposed that all the ordinates signified by the letters from $a$ to $l$ occur between 0 and $\pi/2$. The term written down is the one for the ordinate at $a$ and a similar term must be added for each of the other ordinates from $b$ to $l$.

In any of the formulæ (33) to (44) we can take two of the ordinates undefinitely near together and so obtain the equation of a curve which has a given slope at a given point. By taking three ordinates indefinitely near we can obtain a curve with both a given slope and a given curvature at a given point. Two examples will suffice to illustrate this point.

*Example* 1. Find the odd Class IV function of period $2\pi$ passing through the points $(\delta, S\delta)$ and $\left(\frac{\pi}{2}, H\right)$, when $\delta$ is infinitesimal.

Formula (40) gives us at once

$$f(x) = \frac{\sin x}{\sin \delta} \cdot \frac{\cos 2x - \cos \pi}{\cos 2\delta - \cos \pi} \cdot S\delta$$

$$+ \frac{\sin x}{\sin \pi/2} \cdot \frac{\cos 2x - \cos 2\delta}{\cos \pi - \cos 2\delta} H$$

or

$$f(x) = S \sin x \cos^2 x + H \sin^3 x$$

which may be written

$$f(x) = \frac{S + 3H}{4} \sin x + \frac{S - H}{4} \sin 3x.$$

*Example* 2. Find an even periodic curve of period $2\pi$ through the points (0, 0) $\left(\delta, \frac{R\delta^2}{2}\right)$ and $(\pi, H)$, when $\delta$ is infinitesimal.

Here (37) gives us at once

$$f(x) = \frac{(\cos x - \cos 0)(\cos x - \cos \pi)}{(\cos \delta - \cos 0)(\cos \delta - \cos \pi)} \frac{R\delta^2}{2}$$

$$+ \frac{(\cos x - \cos 0)(\cos x - \cos \delta)}{(\cos \pi - \cos 0)(\cos \pi - \cos \delta)} H$$

or

$$f(x) = \frac{R}{2} \sin^2 x + \frac{H}{4}(1 - \cos x)^2.$$

Problems of this kind, however, are often best solved by writing down a Fourier series, with the least number of the correct kind of harmonics which are seen to be necessary, and then determining the coefficients to satisfy the given conditions.

EXAMPLES.

1. In an odd Class IV function, if 0, $E_1$, $E_2$, $E_3$ are the ordinates at $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ respectively, prove that

$$\begin{aligned} \mathrm{b}_1 &= \tfrac{1}{3}(E_1 + E_2\sqrt{3} + E_3), \\ \mathrm{b}_3 &= \tfrac{1}{3}(2E_1 - E_3), \\ \mathrm{b}_5 &= \tfrac{1}{3}(E_1 - E_2\sqrt{3} + E_3). \end{aligned}$$

2. Show that $\mathrm{b}_1$, $\mathrm{b}_3$ and $\mathrm{b}_5$ in Ex. 1 are respectively,

$$\begin{aligned} \mathrm{b}_1 &= b_1 - b_{11} + b_{13} - b_{23} + b_{25} - \ \ldots \\ \mathrm{b}_3 &= b_3 - b_9 + b_{15} - b_{21} + b_{27} - \ \ldots \\ \mathrm{b}_5 &= b_5 - b_7 + b_{17} + b_{19} + b_{29} - \ \ldots \end{aligned}$$

3. Analyse the odd Class IV function defined by $f(t) = \frac{2t}{\pi}$ from $t = 0$ to $\frac{\pi}{2}$ by using 3, 4, 5, 7 ordinates per quarter-period; and show that in the first case the coefficients obtained satisfy the relations in Ex. 2 and satisfy similar relations in the other cases.

# CHAPTER VI

## THE THEORY OF FOURIER'S SERIES

THE present chapter may be entirely neglected by those whose interest in the subject is purely practical. Nevertheless, we hope we have made the subject so far sufficiently interesting to induce many, who have no intention whatever of making mathematicians of themselves, to peruse this chapter out of an interested curiosity in the subject.

For the possible benefit of a few readers we will first recall one or two theorems on the convergence of infinite series. First, any infinite series $s_1 - s_2 + s_3 - s_4 + \ldots$ in which the terms are alternately positive and negative, and in which each term is numerically less than the one before it and such that the terms decrease ultimately to zero, is convergent. For, writing the series in the form $(s_1 - s_2) + (s_3 - s_4) + \ldots$ we see that the sum is positive; and writing it in the form $s_1 - (s_2 - s_3) - (s_4 - s_5) - \ldots$ we see that it is less than $s_1$. Hence the series is convergent.

Secondly, the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is divergent. For if the reader constructs a series of adjacent rectangles of unit width on the same base and of heights $1, \frac{1}{2}, \frac{1}{3}$, etc., and also draws the curve $y = \frac{1}{x}$ through the vertices so as to lie inside the rectangles he will see at once that the sum of all the rectangles is greater than $\int_1^\infty \frac{dx}{x}$ which equals $\log \infty$ which is infinite.

Thirdly, the series $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots$ is convergent when $n > 1$. For if we draw rectangles as in the last case, and a curve through the vertices lying outside them, we see that the sum of all the terms after the first is less than $\int_1^\infty \frac{dx}{x^n}$, which equals $\frac{1}{n-1}$.

We will next prove that if $s_1, s_2, s_3 \ldots$ be any series of regularly decreasing positive terms decreasing to zero ultimately, no matter how slowly, then the series

$$s_1 \cos \beta + s_2 \cos 2\beta + s_3 \cos 3\beta + \ldots \quad (1)$$

and

$$s_1 \sin \beta + s_2 \sin 2\beta + \ldots \quad (2)$$

are convergent whenever $\beta$ can be expressed in the form $p\pi/q$, where $p$ and $q$ are integers of which $p$ must be odd. Whatever the value of $\beta$ this can be done to any desired degree of accuracy. If, *e.g.*, $\beta = 4\pi/7$, the theorem will hold for $\beta = 4{,}000{,}001\pi/7{,}000{,}002$, so the two series can only be divergent for certain isolated values of $\beta$.

Consider the first $2q$ terms of either of these series and divide these terms into two groups, the positive and the negative terms. Let the sum of the first be denoted by $P_1$ and of the second by $-N_1$. If $s_r \cos r\beta$, say, is any term in $P_1$ then $s_{r+q} \cos (r+q)\beta$, that is, $-s_{r+q} \cos r\beta$ is a term in $N_1$ which by hypothesis is numerically less than the corresponding term $s_r \cos r\beta$ in $P_1$.

Similarly, all the terms in $N_1$ are less than the corresponding terms in $P_1$ and hence $N_1 < P_1$. If now we consider the terms from $s_{2q+1}$ to $s_{4q}$ and denote the sum of the positive and negative terms by $P_2$ and $-N_2$ respectively, we see, by a similar argument, that

$P_2 < N_1$ and that $N_2 < P_2$ and so on. Hence the series of terms,

$$P_1 - N_1 + P_2 - N_2 + P_3 - \ldots,$$

are such that each is numerically less than the preceding and of alternate sign. It follows in this manner that both the series (1) and (2) are convergent.

Writing $\beta + t$ for $\beta$ in (1), we see that

$$\left.\begin{array}{l} s_1 \cos\beta \cos t + s_2 \cos 2\beta \cos 2t + \quad . \\ - s_1 \sin\beta \sin t - s_2 \sin 2\beta \sin 2t - \quad . \quad . \end{array}\right\} \quad (3)$$

is convergent *as a whole*, for all values of $t$. But the first line is an even function of $t$ and the second one an odd function. Now the infinities of an even function cannot everywhere neutralise the infinities of an odd function, for if they neutralised for positive values of $t$ they would re-inforce for negative values. It follows, then, that the series for both the odd and the even functions must be *separately* convergent; and also, that a simular result must hold for the two series,

$$\left.\begin{array}{ll} & s_1 \cos\beta \sin t + s_2 \cos 2\beta \sin 2t + \quad . \quad . \\ \text{and} & s_2 \sin\beta \cos t + s_2 \sin 2\beta \cos 2t + \quad . \quad . \end{array}\right\} \quad (4)$$

obtained by putting $\beta = \beta + t$ in (2).

One obvious exceptional case must now be noted, that is, when $\beta = 0$ or a multiple of $2\pi$ in (1). All the terms are then positive and the series $s_1 + s_2 + s_3 + \ldots$ may be either convergent or divergent, according to the rapidity with which the $s$'s decrease. We conclude from this that either of the series in (3) *may* be infinite when $\beta \pm t$ is zero or a multiple of $2\pi$, but *must* be finite for all other values of $t$.*

Each term of the series (2) is identically zero when $\beta = 0$, but the series may then behave in a very

* If the reader draws, from an origin, radii of lengths $s_1, s_2 \ldots$, in the directions $\beta, 2\beta \ldots$, respectively, it will be nearly (if not quite) obvious to him that the sums of (1) and (2) are necessarily finite for all values of $\beta$ save zero or a multiple of $2\pi$.

peculiar manner which must now be investigated. Let us suppose that $\beta$ is indefinitely small so that we can replace $\sin n\beta$ by $n\beta$, the series then becomes

$$s_1\beta + 2s_2\beta + 3s_3\beta + \ . \ . \ .$$

or

$$\beta[s_1 + 2s_2 + 3s_3 + \ . \ . \ .].$$

Now the series in brackets may be convergent or divergent: if convergent there is no doubt about the sum of (2) being zero when $\beta = 0$, but if divergent the result takes the form $0 \times \infty$ when $\beta = 0$ so that in this case we cannot say anything about the value of the series

$$s_1 \sin \beta + s_2 \sin 2\beta + \ . \ . \ .$$

when $\beta = 0$ except that it is *formally* zero on the strength of the fact that each term is zero.

Let us suppose that $s_n = \frac{1}{n}$, so that the series is

$$\phi(\beta) \equiv \frac{\sin \beta}{1} + \frac{\sin 2\beta}{2} + \frac{\sin 3\beta}{3} + \ . \ . \qquad (5)$$

We may write this

$$\beta\left[\frac{\sin \beta}{\beta} + \frac{\sin 2\beta}{2\beta} + \frac{\sin 3\beta}{3\beta} + \ . \ . \ . + \frac{\sin N\beta}{N\beta} + \ . \ . \ .\right].$$

The student must now recollect that the limit of the sum

$$\beta[f(\beta) + f(2\beta) + \ . \ . \ . + f(N\beta)]$$

when $\beta = 0$ is

$$\int_0^{N\beta} f(x)dx.$$

by the definition of a definite integral.

Hence the limit of the sum of the above series up to the Nth term is

$$\int_0^{N\beta} \frac{\sin x}{x} dx.$$

If now we make $\beta = 0$ and $N = \infty$, we do not get a unique value for the result, but a value which is a function of the product $N\beta$. We conclude that the given series, when $\beta = 0$, may have any value that the integral

$$\int_0^X \frac{\sin x}{x} dx$$

can take. This integral clearly increases with X from $X = 0$ to $X = \pi$, when it assumes the value 1·8519; after then it decreases till $X = 2\pi$, owing to the additional elements being negative, when it takes the value 1·4181; it then goes on oscillating with a diminishing amplitude till $X = \infty$, when it has, as we saw on page 11, the value of $\frac{\pi}{2}$ or 1·5708.

This is the behaviour when $\beta$ has an infinitely small positive value. Since the series in question is an odd function of $\beta$, it behaves in an inverted manner for negative values.

Hence the series (5), when $\beta = 0$, can assume *any value* between $-1·8519$ and $+1·8519$. We shall call the value $\frac{\pi}{2}$ the *principal value* of the series when $\beta$ is $+0$, and $-\frac{\pi}{2}$ the *principal value* when $\beta = -0$, and zero the *formal* value of the series when $\beta = 0$. The principal value corresponds to the value when $N\beta = \infty$, and is thus the value of the series when $\beta$ is an exceedingly small *finite* quantity, or as we may express it, $\phi(+0) = \frac{\pi}{2}$ and $\phi(-0) = -\frac{\pi}{2}$.

We can similarly investigate the behaviour, when $\beta = 0$, of the series

$$s_1 \sin \beta + s_2 \sin 2\beta + \ldots$$

when $s_n = \frac{1}{n^p}$ and $p$ is any positive quantity. The series becomes in this case

$$\beta^p \left\{ \frac{\sin \beta}{\beta^p} + \frac{\sin 2\beta}{2^p \beta^p} + \ldots + \frac{\sin N\beta}{N^p \beta^p} + \ldots \right\}$$

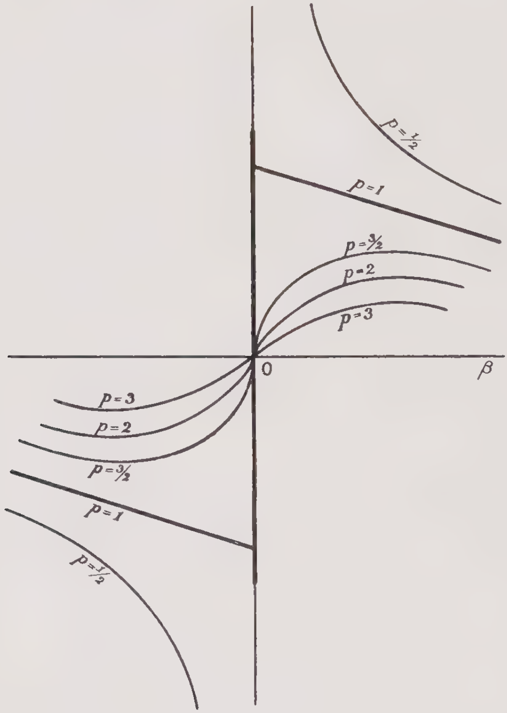


FIG. 20.

which, when $\beta$ tends to zero, becomes

$$\beta^{p-1} \int_0^{N\beta} \frac{\sin x}{x^p} dx \quad . \quad . \quad . \quad . \quad (6)$$

The student familiar with definite integrals will recognise that this integral is finite and convergent

whenever $p < 2$. So if $1 < p < 2$ the result of (6) is zero on account of the factor $\beta^{p-1}$. It can be proved that the result of (6) is zero even if $p > 2$, since the zero of the factor $\beta^{p-1}$ is of higher order than the infinity of the integral owing to the factor

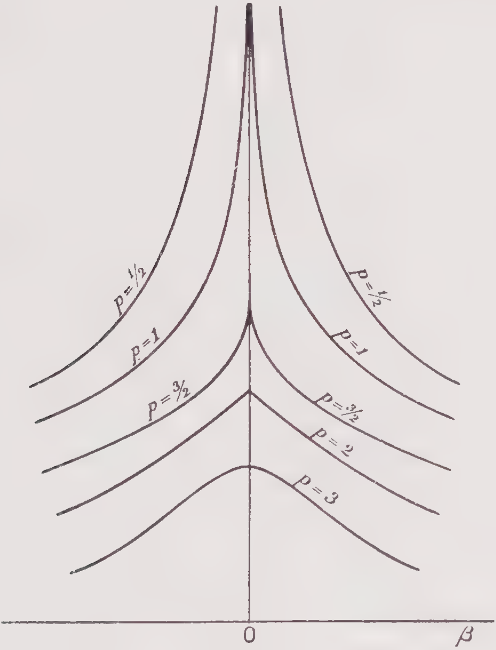


FIG. 21.

$x^p$ in the denominator. If, on the other hand, $p$ lies between 0 and 1 the result of (6) is infinite when $\beta = 0$, unless the range of integration $N\beta$ is infinitely small, owing to the $\beta^{p-1}$ factor which is then infinite. In this case the series can take any value from 0 to $\infty$ when $\beta = +0$, and any value from 0 to $-\infty$ when $\beta = -0$, since it is an odd function.

The appended diagram (Fig. 20) shows qualitatively the behaviour of the series

$$\frac{\sin \beta}{1^p} + \frac{\sin 2\beta}{2^p} + \ldots$$

for the values of $p$ of $\frac{1}{2}$, 1, 3/2, 2 and 3, in the neighbourhood of $\beta = 0$.

Fig. 21, similarly, illustrates qualitatively the behaviour of the series

$$\frac{\cos \beta}{1^p} + \frac{\cos 2\beta}{2^p} + \ldots$$

for the same values of $p$ in the neighbourhood of $\beta = 0$.

In both these diagrams the value of the series is infinite when $p < 1$ and also when $p = 1$ in Fig. 21. The length of the heavy vertical line in Fig. 20 shows the extreme range of undeterminateness of the series (5) when $\beta = 0$. Both arcs, for the value $p = 3/2$, in both these figures, have an infinite slope but a finite radius of curvature at $\beta = 0$; in both diagrams the slope at $\beta = 0$ is finite if $p > 2$ and finite for $p = 2$ in Fig. 21; the curvature at $\beta = 0$ is finite in both cases when $p > 3$ and for $p = 3$ in Fig. 20 if the arcs on the two sides of the origin are considered as separate curves.*

## *The Relation between the Fourier Coefficients and the Properties of a Function.*

It is not necessary that a periodic function $f(t)$ should be everywhere finite for it to be expanded in

* The arcs on the two sides of the origin are necessarily represented by different analytic functions, and each arc has a finite radius of curvature at the origin. This must not be confused with the point of inflection of an analytic curve for which the radius of curvature increases to infinity as the point of inflection is approached.

a Fourier Series; but it is necessary for all the Fourier constants to be finite, and for the higher ones to decrease indefinitely, if the series is to be convergent. The first condition will obviously be satisfied if

$$\int_0^{2\pi} |f(t)| dt \quad . \quad . \quad . \quad . \quad . \quad (7)$$

is finite; for the integral $\int_0^{2\pi} f(t) \cos nt\, dt$ consists of elements none of which are larger than corresponding elements in the first integral; while some are taken positively and some negatively in the second case, and all positively in the first case.

We will next prove that if $f(t)$ is everywhere finite the Fourier constants $a_n$ and $b_n$ cannot be greater, when $n$ is large, than $c/n$, where $c$ is some finite constant. In other words, the Fourier constants, $a_1$, $a_2$, $a_3$ . . ., must decrease, ultimately, at least as fast as the series $c/1$, $c/2$, $c/3$ . . .

The function $f(t)$, in the interval from 0 to $2\pi$, can be divided into a finite number of ranges in each of which range it is of one sign and either constantly increasing or constantly decreasing (or stationary).

Let us consider the value of the integral $\int f(t) \cos nt\, dt$ over any of these ranges when $n$ is fairly large. Over a short range of $\frac{\pi}{n}$ the integrand will be positive and will contribute $s_1$, say, to the integral. Over the next range of $\pi/n$ the integrand will be negative and give $-s_2$, say, and so on up to $s_l$, where $s_l$ corresponds to the last complete half-wave in the range considered. Now the terms

$$s_1 - s_2 + s_3 - \ . \ . \ . \pm s_l$$

are alternately positive and negative and either steadily

increasing or steadily decreasing. In either case we can see by a similar argument to that used on page 119 that the sum cannot exceed either $s_1$ or $s_l$, whichever is the greater. Also, it is clear, neither of these terms can exceed $\frac{2}{\pi} . M . \frac{\pi}{n} = \frac{2M}{n}$, where M is the greatest value of $f(t)$ in the interval considered. To this result there may have to be added the contribution from any fractional half-waves at the beginning and end of the interval, but neither of these can exceed $\frac{2M}{n}$, so that the value of $\int f(t) \cos nt\, dt$ over the range in question cannot possibly exceed $\frac{6M}{n}$. Summing for other ranges, we see that the whole integral from 0 to $2\pi$ cannot exceed $c/n$ where $c$ is some finite constant. The same result obviously holds for the sine integral.

From this important result we can easily prove that if $f(t)$ has no discontinuities in magnitude, and if $f'(t)$ is everywhere finite,* then the Fourier constants $a_n$ and $b_n$ cannot exceed $c/n^2$ where $c$ is some constant.

For

$$a_n + ib_n = \frac{1}{\pi} \int_0^{2\pi} f(t) e^{int}\, dt\,;$$

integrating this by parts, we get

$$a_n + ib_n = \frac{1}{\pi} \left[ \frac{f(t) e^{int}}{in} \right]_0^{2\pi} - \frac{1}{\pi i n} \int_0^{2\pi} f'(t) e^{int}\, dt.$$

If $f(t)$ has no discontinuities in magnitude the first term vanishes, since it is periodic. The integral gives the Fourier constants for the function $f'(t)$ and these,

* The first condition has been stated to prevent misconception: the second condition includes the first.

we have seen, are $< c'/n$ where $c'$ is some constant, since $f'(t)$ is everywhere finite. Hence $a_n + ib_n$, and therefore $a_n$ and $b_n$, are $< c/n^2$ where $c$ is a constant.

In a similar manner, we can prove that if there are no discontinuities in $f(t)$ and $f'(t)$, and if $f''(t)$ be everywhere finite, then the Fourier constants $a_n$ and $b_n$ must be $< c/n^3$ where $c$ is some constant, and so on for discontinuities in the higher differential coefficients.

We will now investigate the order of magnitude of the higher Fourier constants when $f(t)$ becomes infinite at one or more points. If it becomes $\infty$ at more than one point in the range $2\pi$ we can resolve it into a number of functions each of which becomes infinite at only one point. Further, in considering any of these functions, we may take the origin at its point of infinity. We will accordingly suppose that any one of these functions can be represented by

$$f(t) = \frac{k}{t^p} + \phi(t) \text{ or by } f(t) = k \log t + \phi(t),$$

where $\phi(t)$ is under no restrictions save that it is everywhere finite; so that its Fourier constants, when $n$ is large, cannot exceed $c/n$. The Fourier constants for the first term are given by the real and imaginary parts of

$$\frac{k}{\pi}\int_0^{2\pi} \frac{e^{int}}{t^p}\, dt$$

Let us write $nt = x$ so that the limits are now 0 and $2\pi n$; the expression then becomes

$$\frac{k}{\pi\, n^{1-p}} \int_0^{2\pi n} \frac{e^{ix}\, dx}{x^p}.$$

If $p$ lies between 0 and 1 the integral is finite when $n$ is made $\infty$ in the upper limit, so we see that the

K

Fourier constants are of the order of magnitude not greater than $c/n^{1-p}$ when $n$ is large. Compared with this, the constants due to the finite function $\phi(t)$ are ultimately negligible.

Hence we see that the order of magnitude of the higher Fourier constants depends on the nature of the *infinities* of $f(t)$ if it has any, but if not, it depends upon the nature of its *discontinuities in magnitude*; if it has neither, they depend upon the infinities or discontinuities of $f'(t)$ and so on.

It should be noted that if $p < 1$ the integral $\int_0^{2\pi} |f(t)| dt$ is finite; for then

$$\int_0^{2\pi} \frac{k}{t^p} dt = \frac{k(2\pi)^{1-p}}{1-p}.$$

We thus see that whenever the condition that $\int_0^{2\pi} |f(t)| dt$ is finite is satisfied, the Fourier constants decrease indefinitely with $n$, since both these conditions are satisfied at the same time, viz., when $p$ lies between 0 and 1.

### *The Convergency of a Fourier Series.*

The convergency of a Fourier series, representing any function $f(t)$, say the series

$$a_1 \cos t + a_2 \cos 2t + \ldots,$$

only needs to be considered when the series

$$|a_1| + |a_2| + |a_3| + \ldots$$

is divergent, for if this series is convergent the first series is obviously convergent.

It follows from this, and from what we have said above, that it is only the effect of the infinities and discontinuities of magnitude of $f(t)$ on the Fourier

constants that need to be considered here; for discontinuities of slope or curvature only contribute a term of the order $c/n^2$ or $c/n^3$ to the Fourier constants which could not affect the convergency in any way. Let the function $f(t)$ be represented as the sum of several functions each of which has only one point of infinity or of discontinuity of magnitude. When any of these functions are being considered let this point be chosen as the origin.

Now any periodic function of period $2\pi$, which has no infinities, and only one discontinuity, and that at the origin, can only differ from a multiple of the odd function defined by $f(t) = \frac{\pi - t}{2}$ from $t = 0$ to $t = 2\pi$, by a function whose Fourier series is absolutely convergent and therefore need not be considered. But the former function is represented by the series

$$\frac{\sin t}{1} + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \ldots$$

(see page 42), and this *series*, we have seen above (page 120), is convergent for all values except for $t = 0$ (or a multiple of $2\pi$) when it is indeterminate and can assume all values from $-1{\cdot}8519$ to $+1{\cdot}8519$. Since the *function* at $t = 0$ is indeterminate within the range from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, the range of indeterminateness of the *series* is $1{\cdot}179$ times the magnitude of the discontinuity of the *function*. The *formal* sum of the series is zero, which is the point midway between the extreme values of the function at its discontinuity, and this property will obviously be preserved if this series is added to another series which represents any other function with no discontinuity or infinity at $t = 0$.

Similarly, if the function has a single infinity at $t = 0$, and in the neighbourhood of $t = 0$ approximates

to $k/t^p$, where $p$ lies between 0 and 1, we have seen that the series representing it approximates, in its higher terms, to

$$c\left\{\frac{\cos t}{1^{1-p}}+\frac{\cos 2t}{2^{1-p}}+\frac{\cos 3t}{3^{1-p}}+\ldots\right\}$$

or to

$$c'\left\{\frac{\sin t}{1^{1-p}}+\frac{\sin 2t}{2^{1-p}}+\frac{\sin 3t}{3^{1-p}}+\ldots\right\},$$

where $c$ and $c'$ are constants, according to whether the function is odd or even : if it is neither, it can be represented by the sum of both these series. It is, of course, only the behaviour of the higher terms of a series which affects its convergency. Now both these series, we have seen, are convergent for all values of $t$ save $t=0$ (page 120).

When $t=0$, the first series, we saw, was infinite; while the second was indeterminate over an infinite range, which is exactly how the graph representing the *function* behaves in the two cases.

We have thus seen that for any function, with any finite number of discontinuities, or infinities which are such that the integral

$$\int_0^{2\pi}|f(t)|\,dt$$

is finite, the Fourier series *is convergent* and can thus be used to calculate the function.

We may note here that it will easily be found that the series

$$\frac{\cos t}{1}+\frac{\cos 2t}{2}+\frac{\cos 3t}{3}+\ldots,$$

which we have not mentioned represents the artificial periodic function defined by $f(t)=\log\,|\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}t|$. When $t$ is small this becomes $-\log|t|$ approximately;

which is infinite when $t = 0$, but is a lower order of infinity than is represented by $1/t^p$.

Since a periodic function (whether even or odd) which has a single infinity of the order $k/t^p$ at $t = 0$, where $p$ lies between 0 and 1, has its higher Fourier constants approximating to $c/n^{1-p}$ we see, by integrating both the function and the series $(s - 1)$ times, that a series in which the higher Fourier constants approximate to $c/n^{s-p}$ represents a function whose $(s - 1)$th differential coefficient has an infinity of the order $k/t^p$ at $t = 0$. Also from the foregoing results, we see that if the higher Fourier constants approximate to $c/n^s$ where $s$ is an integer, then the $(s - 1)$th differential coefficient will have an ordinary discontinuity in magnitude if it is an odd function and will have an infinity of the nature of $\log |t|$ if it is an even function.

We may remark that the functions in the last paragraph are not necessarily artificial functions though they usually are. For instance the function defined by $y = \sin (t + y)$ is an analytic function. It represents an odd periodic function of period $2\pi$; a distorted sine wave which has an infinite slope at the origin and attains its maximum value of unity when $t = \pi/2 - 1$. Near the origin we have $t = y^3/6$ approximately, or $y = 6^{1/3}t^{1/3}$; from which $\dfrac{dy}{dt} = kt^{-2/3}$, so that the higher Fourier constants of the differential coefficient approximate to $c/n^{1/3}$; and so the higher Fourier constants of the function itself approximate to $c/n^{4/3}$.

The Fourier constants may, of course, decrease much more rapidly than is represented by the law $c/n^s$; $a_n$ and $b_n$ may, for instance, when $n$ is large approximate to $ck^n$ where $k < 1$, as in Problem I of Chapter IV. In this case, if we differentiate the

series $s$ times, $a_n$ and $b_n$ for the differentiated series approximate to $ck^n n^s$; and the ratio of one coefficient to the previous one is $k\left(\frac{n+1}{n}\right)^s$, which, in the limit, when $n$ is infinite, is $k$, so that the series is still convergent. Hence such a series represents a function in which *none* of the differential coefficients are ever infinite or have any discontinuities. Such a function is *necessarily* an analytic function.*

*Proof of Fourier's Theorem by Direct Summation of the Series.*

We have proved in Chapter II that if any periodic function of period $2\pi$ *can* be expanded in the series

$$\frac{a_0}{2} + a_1 \cos t + \ . \ . \ . + b_1 \sin t + \ . \ . \ .$$

then the coefficients *must* have the values there found for them. We also gave later a proof that the expansion was possible (page 23). Just how far the reader will regard these two proofs taken together as constituting a rigorous proof of Fourier's Theorem will depend upon what we may call his mathematical temperament.

Another proof of Fourier's Theorem has virtually been obtained in Chapter V. We there obtained a

* The student familiar with the elements of the Theory of Functions will recognise that the Fourier series in this case will be convergent for complex values of $t$ as well as real ones. If we write $t = u + iv$ in $\frac{a_{n+1}\cos(n+1)t}{a_n \cos nt}$, which is the ratio of one harmonic to the preceding, we find that when $n$ is large it rapidly approaches $a_{n+1}e^{|v|}/a_n$. Hence the series will be convergent so long as $e^{|v|} < k$, *i.e.* so long as $|v| < \log 1/k$.

If the series of Fourier coefficients is still more rapidly convergent, it is possible for the Fourier series to be convergent for all complex values of $t$. Such will be found to be the case when $a_n$ or $b_n$ approximate to $\frac{k^n}{\lfloor n}$ even when $k$ is greater than unity.

sine and cosine series which passed through the tops of a number of equidistant ordinates and we showed (page 78) that when the number of ordinates increased indefinitely, the coefficient of each sine and cosine term moved to a definite limit, and so the finite number of terms became in the limit the infinite Fourier series. The fact that the coefficient of each term approximates to a definite finite limit as the number of ordinates is made infinite, taken in conjunction with the convergency of the series, which we have proved above, is a satisfactory proof of the validity of Fourier's expansion.

If we took any infinite series of different functions, $\phi_1(t)$, $\phi_2(t)$ . . . . we could of course determine the constants in the expression

$$c_1\phi_1(t) + c_2\phi_2(t) + \ . \ . \ . \ + c_n\phi_n(t) \qquad (8)$$

so as to make it pass through the tops of any $n$ given ordinates; but it would not in general happen that the $c$'s tended to any definite limit as $n$ was increased to infinity. Nor would it generally happen that the graph represented by (8) would, when the ordinates were sufficiently close together, approximate to the straight lines joining the tops of successive ordinates. In general, the curve would make a half oscillation in running from the top of one ordinate to the next; a fact which is consistent with, and explains, the fact that the expression (8) has no definite limit when $n$ tends to infinity. The fact that the sine-and-cosine series on the contrary approaches a limit, means that the graph represented by it approaches a limit, and this limit cannot be anything else than the limit of the straight lines joining the tops of consecutive ordinates.

We will now sum directly the whole of the first $n$ harmonics of Fourier's expansion including the constant term. We have

$$f(t) = \frac{a_0}{2} + a_1 \cos t + \ldots$$
$$+ b_1 \sin t + \ldots$$

where $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t) \cos nt\, dt$$

and $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t) \sin nt\, dt.$$

Let us use the letter $x$ for the variable in these integrals and, since the starting-point of the range is immaterial so long as the range is $2\pi$, make the limits $t - \pi$ and $t + \pi$; so that

$$a_n = \frac{1}{\pi}\int_{t-\pi}^{t+\pi} f(x) \cos nx dx.$$

The apparent dependence of $a_n$ on $t$ in this result is illusory, for, as we have just said, the value of $t$ does not affect the integral.

Replacing $b_n$ by a similar expression and adding together the two terms constituting the $n$th harmonic, we have

$$a_n \cos nt + b_n \sin nt = \frac{1}{\pi}\int_{t-\pi}^{t+\pi} f(x) \cos n(x - t) dx.$$

On writing $x = u + t$ this becomes

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t + u) \cos nu\, du.$$

Hence the sum, $S_n$, of the first $n$ harmonics plus the constant term is

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t + u)\left\{\frac{1}{2} + \cos u + \cos 2u + \ldots + \cos nu\right\}du.$$

Replacing each cosine term by its exponential value, we easily find that the value of the series is $\dfrac{\sin (n + \frac{1}{2})u}{2 \sin \frac{1}{2}u}$.

Hence we have

$$S_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t+u)\,\frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\,du.$$

Now change the variable again by writing $(n+\frac{1}{2})u=v$, and for brevity write $n'$ for $n+\frac{1}{2}$. We then get

$$S_n = \frac{1}{\pi}\int_{-n'\pi}^{n'\pi} f\left(t+\frac{v}{n'}\right)\cdot\frac{\sin v}{2n'\sin\dfrac{v}{2n'}}\,.\,dv\,. \qquad (9)$$

If $n'$ is large, $f\left(t+\dfrac{v}{n'}\right)$, for moderate values of $v$ say for $v < V$, is sensibly equal to $f(t)$ unless $f'(t)$ be very great; also $2n'\sin\dfrac{v}{2n'}$ is then sensibly equal to $v$, hence, taken over for the range from $-V$ to $V$, the integral is sensibly equal to

$$\frac{f(t)}{\pi}\int_{-V}^{V}\frac{\sin v}{v}\,dv \qquad (10)$$

Now this converges when $V = \infty$, the contribution from large values of $v$ being really less than seems to be the case, since each positive half-wave is very nearly neutralised by the following negative half, when $v$ is large. If V is moderately large, then, the value of this integral is sensibly that of

$$\int_{-\infty}^{\infty}\frac{\sin v}{v}\,dv,$$

which we have seen (page 11) is $\pi$. There is clearly no difficulty in choosing a large finite value of V which will make the integral in (10) as near to $\pi$ as we choose and *then* in taking $n'$ much larger than V so that the

approximations (of $2n' \sin \frac{v}{2n'} = v$ and $f\left(t + \frac{v}{n'}\right) = f(t)$) in (9) are as accurate as we wish them.

We conclude, then, that the limit of $S_n$, when $n$ is infinite, is $f(t)$ provided $f'(t)$ is finite. This furnishes a rigorous proof of Fourier's expansion.

If the function $f(t)$ has a discontinuity in magnitude at the particular value of $t$ considered, the result (9) breaks up into two, giving

$$S_n = \frac{1}{\pi}\int_0^{n'\pi} f\left(t + \frac{v}{n'}\right) \frac{\sin v}{2n' \sin \frac{v}{2n'}}\, dv$$

$$+ \frac{1}{\pi}\int_{-n'\pi}^{0} f\left(t - \frac{v}{n'}\right) \frac{\sin v}{2n' \sin \frac{v}{2n'}}\, dv.$$

As previously, the first integral becomes in the limit

$$\frac{f(t+0)}{\pi}\int_0^{\infty} \frac{\sin v}{v}\, dv = \tfrac{1}{2} f(t+0),$$

and the second integral becomes likewise

$$\frac{f(t-0)}{\pi}\int_{-\infty}^{0} \frac{\sin v}{v}\, dv = \tfrac{1}{2} f(t-0);$$

so that the sum to infinity of the Fourier series is

$$\tfrac{1}{2}\{f(t+0) + f(t-0)\},$$

at a point of discontinuity, and $f(t)$ at other points.*

* The student with a little mathematical courage will probably obtain this result by boldly putting $n' = \infty$ in (9) when it becomes

$$S_\infty = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t) \frac{\sin v}{v}\, dv = \frac{\pi}{\pi} f(t),$$

and this is not so reprehensible as some authors would have us believe. It is necessary, however, before writing $\infty$, or any other particular

At a point of discontinuity this investigation clearly obtains what we have called the *formal* sum of the series; and we have seen before that at such points *this* sum of the series is midway between $f(t+0)$ and $f(t-0)$.*

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value for a variable, to make sure that the expression assumes a single definite value when this is done. A case in point is the series

$$\sin 1/t + \tfrac{1}{2}\sin 2/t + \tfrac{1}{3}\sin 3/t + \ldots.$$

As $t$ increases towards $+\infty$, this series, as we have seen, approaches the limit of $+\frac{\pi}{2}$; while when $t$ is negative and moves towards $-\infty$ the series approaches the limit of $-\frac{\pi}{2}$ while actually *at* $t=\pm\infty$ the series is indeterminate and can assume any value from $-1{\cdot}8519$ to $+1{\cdot}8519$. Such behaviour, however, is not difficult to detect, and in the case of infinite series, is always bound up with the fact that *at* such a point the series does not begin to converge till after an infinite number of terms.

* It is clear also that the limit of (9) when $n'$ is $\infty$ will only be

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{\sin v}{v}\,dv \text{ or } f(t)$$

provided that $f\left(t+\frac{v}{n'}\right)$ is sensibly constant for all values of $v$ below a reasonably large quantity, say V, when $n'$ is very large. This is generally expressed by saying that $f(t)$ must not have an infinite number of maxima and minima or discontinuities in a finite space in the neighbourhood of the value of $t$ considered; for if so, $f\left(t+\frac{v}{n'}\right)$ could fluctuate in magnitude during a finite change in $v$ when $n'$ was infinite: and so we could tell nothing about the value of the integral

$$\int f\left(t+\frac{v}{n'}\right)\frac{\sin v}{v}\,dv.$$

This is the reason why so many books dealing with Fourier's series continually repeat the condition that the function must not have an infinite number of maxima or minima. We have generally omitted specifying this condition, since no practical function ever does behave in such a manner. Such behaviour is exclusively confined to functions invented by mathematicians for the sake of causing trouble.

## EXAMPLES.

1. By putting $t=0$ in the formula (21) of page 30, and by replacing the $a$'s by their integral values, prove that

$$\tfrac{1}{2}f(0)+f\left(\frac{2\pi}{n}\right)+f\left(\frac{4\pi}{n}\right)+\ldots+\tfrac{1}{2}f(2\pi)=\frac{n}{2\pi}\int_0^{2\pi}f(t)dt$$

$$+\frac{n}{\pi}\int_0^{2\pi}f(t)\cos ntdt+\frac{n}{\pi}\int_0^{2\pi}f(t)\cos 2ntdt+\ldots$$

[The reason for writing $\frac{1}{2}f(0)+\frac{1}{2}f(2\pi)$ instead of *either* $f(0)$ *or* $f(2\pi)$ is in case the *periodic function* $f(t)$ has a discontinuity at $t=0$ which will of course be the case if $f(2\pi)\neq f(0)$.]

2. Prove, by repeated integration by parts, that

$$\int_0^{2\pi}f(t)\cos ntdt=\frac{f'(2\pi)-f'(0)}{n^2}-\frac{f'''(2\pi)-f'''(0)}{n^4}+\ldots$$

provided that all the differential coefficients are finite and the series is convergent.

3. Prove, by substituting the result of Ex. 2 in Ex. 1, that

$$\int_0^{2\pi}f(t)dt=\frac{2\pi}{n}\left\{\tfrac{1}{2}f(0)+f\left(\frac{2\pi}{n}\right)+\ldots+\tfrac{1}{2}f(2\pi)\right\}$$

$$-2\frac{f'(2\pi)-f'(0)}{n^2}\left\{1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right\}$$

$$+2\frac{f'''(2\pi)-f'''(0)}{n^4}\left\{1+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\right\}-\ldots$$

The numerical series in brackets are known to have the values $\frac{\pi^2}{6}$, $\frac{\pi^4}{90}$, $\frac{\pi^6}{945}$, $\frac{\pi^8}{9450}$, $\frac{\pi^{10}}{93555}$, . . . respectively.

4. By changing the range of integration in Ex. 3 from $2\pi$ to $nh$ prove that

$$\int_0^{nh}f(t)dt=h\{\tfrac{1}{2}f(0)+f(h)+f(2h)+\ldots+\tfrac{1}{2}f(nh)\}$$

$$-\frac{h^2}{2\,\underline{|3}}\left\{f'(nh)-f'(0)\right\}+\frac{h^4}{6\,\underline{|5}}\left\{f('''nh)-f'''(0)\right\}$$

$$-\frac{h^6}{6\,\underline{|7}}\left\{f^{\mathrm{v}}(nh)-f^{\mathrm{v}}(0)\right\}+\frac{3h^8}{10\,\underline{|9}}\left\{f^{\mathrm{vii}}(nh)-f^{\mathrm{vii}}(0)\right\}$$

$$-\frac{5h^{10}}{6\,\underline{|11}}\left\{\cdot\;\cdot\right\}\ldots$$

provided the series is convergent. The series can often be used when it is not convergent, provided the terms decrease to a small value before

they finally increase, for it may be proved that in many cases the error made by stopping at any term is less than that term.

5. By taking the limits of integration as $x$ and $\infty$ prove that the result of Ex. 4 becomes

$$\int_x^\infty f(t)dt = h\{\tfrac{1}{2}f(x) + f(x+h) + f(x+2h) + \ldots\}$$
$$+ \frac{h^2}{2\,\lfloor 3}f'(x) - \frac{h^4}{6\,\lfloor 5}f'''(x) + \frac{h^6}{6\,\lfloor 7}f^{\mathrm{v}}(x) - \frac{3h^8}{10\,\lfloor 9}f^{\mathrm{vii}}(x) + \ldots$$

supposing the integral convergent.

6. By taking $f(t) = \dfrac{1}{t}$ in Ex. 4 and the limits of integration as $m$ and $n$, which are integers, prove that

$$\log\frac{m}{n} = \frac{1}{2n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{m-1} + \frac{1}{2m}$$
$$- \frac{1}{12}\left(\frac{1}{n^2} - \frac{1}{m^2}\right) + \frac{1}{120}\left(\frac{1}{n^4} - \frac{1}{m^4}\right) - \frac{1}{252}\left(\frac{1}{n^6} - \frac{1}{m^6}\right)$$
$$+ \frac{1}{240}\left(\frac{1}{n^8} - \frac{1}{m^8}\right) \ldots$$

Calculate $\log_e 2$ to seven decimal places by taking $n = 10$ and $m = 20$. [Ans. ·6931472.]

7. Prove, from the result of Ex. 6, that the limit when $m$ is infinite of $\dfrac{1}{1} + \dfrac{1}{2} + \dfrac{1}{3} + \ldots\ldots + \dfrac{1}{m} - \log m$ is a constant. Show, by taking $n = 4$, that this constant is ·577215 approximately.

8. By taking $f(t) = \log t$ and the limits $m$ and $n$ as integers, show that

$$m\log m/e - n\log n/e = \tfrac{1}{2}\log n + \log(n+1) + \ldots$$
$$+ \log(m-1) + \tfrac{1}{2}\log m - \frac{1}{12}\left\{\frac{1}{m} - \frac{1}{n}\right\} + \frac{1}{360}\left\{\frac{1}{m^3} - \frac{1}{n^3}\right\}$$
$$- \frac{1}{1260}\left\{\frac{1}{m^5} - \frac{1}{n^5}\right\} + \ldots$$

Regarding $m$ as a variable and $n$ as a constant, show that this result can be written

$$\log \lfloor m = \text{Const.} + (m + \tfrac{1}{2})\log m - m + \frac{1}{12m} - \frac{1}{360m^3}$$
$$+ \frac{1}{1260m^5} - \frac{1}{1680m^7} + \frac{1}{1188m^9} - \ldots$$

By taking $m = 10$, show that the constant is ·91893856 . . . It can be shown theoretically that this constant is $\tfrac{1}{2}\log_e 2\pi$.

Deduce from the last result, that when $m$ is large, $\underline{|m}$ approximates to a ratio of equality with

$$\sqrt{2\pi m}\left(\frac{m}{e}\right)^m.$$

9. From Ex. 5, page 73, obtain the following results

$$-\int_0^x \log \sin x\, dx = x \log 2 + \tfrac{1}{2}\left\{\frac{\sin 2x}{1^2} + \frac{\sin 4x}{2^2} + \frac{\sin 6x}{3^2} + \dots\right\}$$

when $0 < x < \pi$;

$$-\int_0^x \log \cos x\, dx = x \log 2 - \tfrac{1}{2}\left\{\frac{\sin 2x}{1^2} - \frac{\sin 4x}{2^2} + \frac{\sin 6x}{3^2} - \dots\right\}$$

when $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

and

$$-\int_0^x \log \tan x\, dx = \frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \frac{\sin 10x}{5^2} + \dots$$

when $0 < x < \dfrac{\pi}{2}$.

10. Prove, by differentiation or by replacing the sines and cosines by their power series, that the curve represented by the series

$$\frac{\cos \beta}{1^p} + \frac{\cos 2\beta}{2^p} + \frac{\cos 3\beta}{3^p} + \dots$$

is finite when $\beta = 0$ if $p > 1$ and has a finite radius of curvature at $\beta = 0$ if $p > 3$. Also prove that the series

$$\frac{\sin \beta}{1^p} + \frac{\sin 2\beta}{2^p} + \dots$$

has a finite slope at the origin if $p > 2$ and an ordinary point of inflection at the origin if $p > 4$.

11. If $\varphi(x) = 0$ when $x = 0$, prove that the series,

$$\frac{\varphi(x)}{1} + \frac{\varphi(2x)}{2} + \frac{\varphi(3x)}{3} + \dots$$

can have any value when $x = 0$ which the integral

$$\int_0^X \frac{\varphi(x)}{x} dx$$

can assume, where X is any positive quantity.

# CHAPTER VII

## FOURIER'S INTEGRAL THEOREM *

Fourier's theorem, for a periodic function $f(t)$ of period T, is that $f(t)$ can be expanded in the form (see page 29)

$$f(t) = \frac{a_0}{2} + a_1 \cos \frac{2\pi t}{T} + a_2 \cos \frac{4\pi t}{T} + \ldots$$

$$+ b_1 \sin \frac{2\pi t}{T} + b_2 \sin \frac{4\pi t}{T} + \ldots \qquad (1)$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \cos \frac{2n\pi t}{T} dt \qquad (2)$$

and

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \sin \frac{2n\pi t}{T} dt \qquad (3)$$

If we change the variable under the sign of integration from $t$ to $\lambda$, we may multiply (2) by $\cos \frac{2n\pi t}{T}$ and (3) by $\sin \frac{2n\pi t}{T}$ and put these terms under the sign of integration; since they are mere constants during the integration with respect to $\lambda$. The two terms $a_n \cos \frac{2n\pi t}{T} + b_n \sin \frac{2n\pi t}{T}$ constituting the $n$th

* This chapter is intended for the student of physics; it has no engineering applications; but we hope it will not on that account be found uninteresting to all engineering students.

harmonic then become

$$\frac{2}{T}\int_{-T/2}^{T/2} f(\lambda) \cos \frac{2n\pi(\lambda - t)}{T} d\lambda.$$

We will write $\frac{2\pi}{T} = \sigma$ and now suppose T increases without limit so that $\sigma$ becomes indefinitely small; the above expression then becomes

$$\frac{\sigma}{\pi}\int_{-\pi/\sigma}^{\pi/\sigma} f(\lambda) \cos n\sigma(\lambda - t) d\lambda.$$

Now replace all the harmonics in (1) by expressions of this type; we get

$$f(t) = \frac{\sigma}{\pi}\int_{-\pi/\sigma}^{\pi/\sigma} \tfrac{1}{2}f(\lambda)d\lambda + \frac{\sigma}{\pi}\int_{-\pi/\sigma}^{\pi/\sigma} f(\lambda) \cos \sigma(\lambda - t)d\lambda$$
$$+ \frac{\sigma}{\pi}\int_{-\pi/\sigma}^{\pi/\sigma} f(\lambda) \cos 2\sigma(\lambda - t)d\lambda + \ldots$$

Now we know that if $F(x)$ is any function of $x$, the limit of $\sigma\{F(0) + F(\sigma) + F(2\sigma) + \ldots + F(x)\}$ when $\sigma = 0$ is $\int_0^x F(x)dx$; and, since each single term is indefinitely small in the limit, it makes no difference whether the first term is $\sigma F(0)$ or $\frac{1}{2}\sigma F(0)$. Applying this to the above case, we see that if we write $\sigma = d\alpha$ * and replace the limits $\pm\pi/\sigma$ by $\pm\infty$, which they then become, we have

$$f(t) = \frac{1}{\pi}\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} f(\lambda) \cos \alpha(\lambda - t)d\lambda \quad . \quad . \quad (4)$$

This result is known as *Fourier's Integral Theorem*.

* This is quite a different use for $\alpha$ than that in the earlier part of this book: we do it for the sake of uniformity with other writers.

The theorem, of course, only holds provided the integrals are convergent. The integral

$$\int_{-\infty}^{\infty} f(\lambda) \cos \alpha(\lambda - t) d\lambda . \quad . \quad . \quad . \quad (5)$$

will necessarily be convergent provided

$$\int_{-\infty}^{\infty} | f(\lambda) | \, d\lambda$$

is convergent. We will accordingly suppose that this condition is satisfied. This requires that $f(\lambda)$ should decrease faster than $1/\lambda$ when $\lambda$ is large, for both positive and negative values. The student will also see, in a similar manner to that adopted in the last chapter, that if $f(\lambda)$ is finite everywhere, but has discontinuities in magnitude, each integral with respect to $\lambda$ in the equation

$$f(t) = \frac{1}{\pi}\int_{0}^{\infty} \Big[ \cos \alpha t \int_{-\infty}^{\infty} f(\lambda) \cos \alpha\lambda \, d\lambda$$

$$+ \sin \alpha t \int_{-\infty}^{\infty} f(\lambda) \sin \alpha\lambda \, d\lambda \Big] d\alpha, \quad . \quad . \quad . \quad (6)$$

in which form (4) may clearly be written, will tend to proportionality with $1/\alpha$ when $\alpha$ is large; and if $f(\lambda)$ has discontinuities in slope but not in magnitude, each integral will likewise tend to proportionality with $1/\alpha^2$, and so on. The integration with respect to $\alpha$ is always convergent when $\int_{-\infty}^{\infty} | f(\lambda) | \, d\lambda$ is convergent owing to the $\cos \alpha t$ or $\sin \alpha t$ factors, which virtually make the integral a series whose terms are alternately positive and negative such that, when $\alpha$ is large, each term is less than the one before it.

Like Fourier's series, Fourier's Integral Theorem applies both to analytic or artificial functions, subject to the integral

$$\int_{-\infty}^{\infty} |f(\lambda)|\, d\lambda$$

being convergent.

We can have functions of two distinct types; either functions gradually increasing from zero at $t = -\infty$ and decaying to zero again as $t$ tends $t + \infty$; or those which are zero from $t = -\infty$ to $t = t_1$, say, when they suddenly become finite and remain finite from $t = t_1$ to $t = t_2$, say, when they suddenly become zero again and remain zero up to $t = \infty$. A function of either of these types is conveniently called a "disturbance" or a "Pulse." It is important to bear in mind that it must necessarily be defined from $t = -\infty$ to $t = +\infty$ or Fourier's Integral Theorem cannot apply to it. If we take the limits of $\lambda$ as $t_1$ and $t_2$ in Fourier's Integral Theorem, instead of $-\infty$ and $+\infty$, we are simply applying the theorem to the pulse which is equal to $f(t)$ from $t = t_1$ to $t = t_2$ and zero outside these limits.

Fourier's integral theorem takes a simpler form when $f(t)$ is either an odd or an even function, as might have been expected from the behaviour of Fourier's series. If, for instance, $f(t)$ is an odd function of $t$, the first integral with respect to $\lambda$ in (6) vanishes; since $f(\lambda)$ is an odd and $\cos \alpha\lambda$ an even function of $\lambda$ and the product is odd: also $f(\lambda) \sin \alpha\lambda$ being an even function, the integral from $-\infty$ to $+\infty$ is equal to twice the integral from 0 to $\infty$.

Hence, if $f(t)$ is an odd function of $t$, we have

$$f(t) = \frac{2}{\pi}\int_0^{\infty} \sin \alpha t \left[ \int_0^{\infty} f(\lambda) \sin \alpha\lambda\, d\lambda \right] d\alpha \quad . \quad (7)$$

Similarly, if $f(t)$ is an even function of $t$ we have

$$f(t) = \frac{2}{\pi}\int_0^\infty \cos \alpha t\left[\int_0^\infty f(\lambda)\cos \alpha\lambda\, d\lambda\right] d\alpha \quad . \quad (8)$$

If we write

$$\int_0^\infty f(\lambda)\sin \alpha\lambda\, d\lambda = \phi(\alpha),$$

and

$$\int_0^\infty f(\lambda)\cos \alpha\lambda\, d\lambda = \psi(\alpha),$$

these results may be written in the following form :

$$\left.\begin{array}{ll}\text{that if} & \displaystyle\int_0^\infty f(\lambda)\sin \alpha\lambda\, d\lambda = \phi(\alpha) \\ \text{then} & \displaystyle\int_0^\infty \phi(\alpha)\sin \alpha t\, d\alpha = \frac{\pi}{2} f(t)\end{array}\right\} \quad . \quad . \quad . \quad (9);$$

$$\left.\begin{array}{ll}\text{and if} & \displaystyle\int_0^\infty f(\lambda)\cos \alpha\lambda\, d\lambda = \psi(\alpha) \\ \text{then} & \displaystyle\int_0^\infty \psi(\alpha)\cos \alpha t\, d\alpha = \frac{\pi}{2} f(t)\end{array}\right\} \quad . \quad . \quad . \quad (10)$$

In the first of these $f(\lambda)$ is odd, and in the second it is even; and, since we see that the relation between $f$ and $\phi$ and $f$ and $\psi$ is reciprocal, $\left(\text{save for the factor } \frac{\pi}{2}\right)$ we see that, *if we wish to interpret $\phi$ and $\psi$ for negative values* of $\alpha$, $\phi$ is to be taken as odd and $\psi$ as even.

As an example, let $f(t) = e^{-ct}$ from $t = 0$ to $t + \infty$. Then we have by (9)

$$\int_0^\infty e^{-c\lambda}\sin \alpha\lambda\, d\lambda = \phi(\alpha),$$

but this integral is $\dfrac{\alpha}{c^2 + \alpha^2}$;

whence we infer that

$$\int_0^\infty \frac{a}{c^2 + a^2} \sin at \, da = \frac{\pi}{2} e^{-ct};$$

or, changing symbols, that

$$\int_0^\infty \frac{x \sin mx}{c^2 + x^2} dx = \frac{\pi}{2} e^{-cm}.$$

Similarly by (10), since

$$\int_0^\infty e^{-c\lambda} \cos a\lambda \, d\lambda = \frac{c}{c^2 + a^2};$$

we have

$$\int_0^\infty \frac{c}{c^2 + a^2} \cos at \, da = \frac{\pi}{2} e^{-ct};$$

or, in other symbols,

$$\int_0^\infty \frac{\cos mx}{c^2 + x^2} dx = \frac{\pi}{2c} e^{-cm}.$$

Fourier's integral theorem in the forms (9) and (10) thus becomes a useful theorem in the integral calculus.

The way in which Fourier's integral theorem performs the "harmonic analysis" of a pulse can be seen by the following considerations. The ratio of the period of the $n$th harmonic to the $(n + 1)$th is $\frac{n+1}{n}$, which tends to unity when $n$ is infinite. Also, when the period is made infinite, all harmonics of *finite* period are necessarily harmonics of an *infinite* order; they are thus infinitely close together, or in other words, there are an infinite number of harmonics whose periods lie between two given finite limits. Now the equation (6) can be written

$$f(t) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(\lambda)\cos\alpha\lambda\, d\lambda\right]\cos\alpha t\, d\alpha$$
$$+ \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(\lambda)\sin\alpha\lambda\, d\lambda\right]\sin\alpha t\, d\alpha$$

Dropping the integration with respect to $\alpha$, we see that

$$\frac{1}{\pi}\left[\int_{-\infty}^\infty f(\lambda)\cos\alpha\lambda\, d\lambda\right]\cos\alpha t\, d\alpha$$
$$+ \frac{1}{\pi}\left[\int_{-\infty}^\infty f(\lambda)\sin\alpha\lambda\, d\lambda\right]\sin\alpha t\, d\alpha \quad . \quad . \quad . \quad (11)$$

is the part of $f(t)$ which consists of harmonics whose periods lie between $\frac{2\pi}{\alpha}$ and $\frac{2\pi}{\alpha + d\alpha}$, or whose frequencies lie between $\frac{\alpha}{2\pi}$ and $\frac{\alpha + d\alpha}{2\pi}$. If we denote the integrals within square brackets by P and Q respectively, (11) can be written

$$\left(\frac{P d\alpha}{\pi}\right)\cos\alpha t + \left(\frac{Q d\alpha}{\pi}\right)\sin\alpha t.$$

Here $\alpha$ is not quite constant but varies over the small range $d\alpha$ from $\alpha$ to $\alpha + d\alpha$, so we see that the amplitude of the $\cos\alpha t$ term at $\alpha$ *may be regarded as* $P/\pi$ *per unit range of* $\alpha$ or $2P$ per unit range of frequency. Similarly, the amplitude of the $\sin\alpha t$ term at $\alpha$ may be regarded as $Q/\pi$ per unit range of $\alpha$, or $2Q$ per unit range of frequency; and the amplitude of the complete harmonic as $2\sqrt{P^2 + Q^2}$ per unit range of frequency.

It will now, we think, be best if we adopt the language of that branch of physics in which this subject has its chief applications. The pulse is generally supposed to be some disturbance travelling through a medium; it may be, say, an air pulse caused by an

explosion, or an electromagnetic pulse due to the stoppage of an electron by collision. In the first place we should take $f(t)$ as proportional to the condensation in the medium at any given point; and in the second case to the electric (or magnetic) field at the point considered. In either case the energy per unit volume of the medium at time $t$ is proportional to $f(t)^2$, and so we may take $\int_{-\infty}^{\infty} f(t)^2 dt$ as a measure of the energy of the whole pulse.*

Now on page 34, in finding an expression for the mean square of the value of a periodic function in terms of the coefficients, we were really expressing the fact that the energy in the function was equal to the sum of the energies in each of its harmonics. An exactly similar result holds in the present case, which we will now obtain. Taking the equation

$$f(t) = \frac{1}{\pi}\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} f(\lambda)\,(\cos \alpha\lambda \cos \alpha t + \sin \alpha\lambda \sin \alpha t)\,d\lambda$$

or, with our previous notation,

$$f(t) = \frac{1}{\pi}\int_0^{\infty} d\alpha\,(\mathrm{P} \cos \alpha t + \mathrm{Q} \sin \alpha t)\,;$$

let us multiply both sides by $f(t)dt$ and integrate from $-\infty$ to $+\infty$. Remembering that P and Q are functions of $\alpha$ only, we have

$$\int_{-\infty}^{\infty} f(t)^2 dt = \frac{1}{\pi}\int_0^{\infty} d\alpha \left[\mathrm{P}\int_{-\infty}^{\infty} f(t) \cos \alpha t\, dt + \mathrm{Q}\int_{-\infty}^{\infty} f(t) \sin \alpha t\, dt\right]$$

or
$$\int_{-\infty}^{\infty} f(t)^2 dt = \frac{1}{\pi}\int_0^{\infty} (\mathrm{P}^2 + \mathrm{Q}^2)\,d\alpha.$$

* There is no need to introduce the factor $\frac{1}{2}$, which the student may be tempted to supply, as we have dropped other constant factors depending on the properties of the medium.

This shows that the total energy may be divided up among the different values of $\alpha$, the energy lying between $\alpha$ and $\alpha + d\alpha$ being $\frac{P^2 + Q^2}{\pi} \cdot d\alpha$; that is, the energy per unit range of $\alpha$ at $\alpha$ is $\frac{P^2 + Q^2}{\pi}$; or, the energy per unit range of frequency at the frequency $\frac{\alpha}{2\pi}$, is $2(P^2 + Q^2)$. Since the energy of an oscillation is proportional to the square of the amplitude, this result is consistent with the one arrived at on p. 149.

When the different frequencies in a pulse are separated in any manner the result is called the "spectrum" of the pulse. The light given out by a piece of red-hot metal or carbon, for instance, consists of the "pulses" produced by the collision of the electrons with one another or with the atoms of the substance. There is not necessarily any periodic motion whatever associated with the origin of the light from an incandescent solid body. A glass prism or a diffraction grating analyses this succession of pulses into its constituent harmonics in exactly the same manner as Fourier's integral theorem analyses it. The fact that the spectrum, in this case, shows the presence of harmonics of all frequencies within the range of the instrument, shows not only that the origin of the light is in pulses, each containing all frequencies in itself, but that these succeed one another at irregular intervals. For a succession of exactly similar pulses at exactly equal intervals of time would obviously constitute a periodic disturbance, from which all harmonics would be absent except those whose periods were sub-multiples of the given interval; hence the spectrum would be a "bright line" spectrum and not a continuous spectrum. The distribution of

energy in a succession of similar pulses can only be the same as the distribution in each pulse if they follow one another at perfectly random intervals of time.

If we make a slit in the screen on which the spectrum is focussed we can let through any desired small range of frequency at any point, and by absorbing these rays we can measure the energy in this small region of the spectrum. In this manner we can get experimentally the distribution of energy in the spectrum.*

We have seen above that if we knew the form of the pulses we could calculate this distribution of energy. We shall now show that, knowing the distribution of energy, we can find the shape of an odd pulse, and also of an even one, which would give such assigned distribution of energy in the spectrum.†

Let $\{F(\alpha)\}^2 d\alpha$ be the energy in the spectrum between $\alpha$ and $\alpha + d\alpha$. Then we have $\pi F(\alpha)^2 = P^2 + Q^2$. Accordingly, if the pulse is even so that $Q = 0$, we have $\sqrt{\pi}F(\alpha) = P = 2\int_0^\infty f(\lambda)\cos\alpha\lambda\, d\lambda$, where $f(t)$ is the form of the pulse.

Equations (10) at once give us the result that

$$\int_0^\infty \frac{\sqrt{\pi}}{2} F(\alpha)\cos\alpha t\, dt = \frac{\pi}{2} f(t)$$

or
$$f(t) = \frac{1}{\sqrt{\pi}}\int_0^\infty F(\alpha)\cos\alpha t\, d\alpha.$$

* It is, of course, necessary to correct the result for the light lost by reflection or absorption in the prism or grating.

† The student familiar with radiation from electrons will recognise that if an electron is projected against another and is brought to rest by the interaction and repelled back along the line of advance, the pulse will be an even one; while if it had passed on without being brought to rest the pulse would have been an odd one. This is easily seen, since the disturbance at a great distance is proportional to the acceleration of the electron.

Similarly, for an odd pulse its form would be given by

$$f(t) = \frac{1}{\sqrt{\pi}}\int_0^\infty F(\alpha) \sin \alpha t \, d\alpha.$$

*Example.* Lord Rayleigh proposed the expression $C^2\alpha^2e^{-2c\alpha}d\alpha$ for the energy in the spectrum between the frequencies $\frac{\alpha}{2\pi}$ and $\frac{\alpha + d\alpha}{2\pi}$, the constants C and $c$ depending upon the temperature of the body emitting the light. Here, $F(\alpha) = C\alpha e^{-c\alpha}$, so that an even pulse to give this distribution would be given by,

$$f_0(t) = \frac{C}{\sqrt{\pi}}\int_0^\infty \alpha e^{-c\alpha} \cos \alpha t \, d\alpha$$

and an odd pulse by,

$$f_1(t) = \frac{C}{\sqrt{\pi}}\int_0^\infty \alpha e^{-c\alpha} \sin \alpha t \, d\alpha.$$

Multiplying the second by $\sqrt{-1}$ and adding it to the first we get

$$f_0(t) + i f_1(t) = \frac{C}{\sqrt{\pi}}\int_0^\infty \alpha e^{(it-c)\alpha} d\alpha = \frac{C}{\sqrt{\pi}(it-c)^2} = \frac{C(it+c)^2}{\sqrt{\pi}(t^2+c^2)^2}.$$

And so

$$f_0(t) = \frac{C(c^2 - t^2)}{\sqrt{\pi}(c^2 + t^2)^2}$$

and

$$f_1(t) = \frac{2Cct}{\sqrt{\pi}(c^2 + t^2)^2}.$$

Similarly, we could find the shape of the pulses to produce any other law of distribution of energy in the spectrum, such as Planck's, whose formula agrees much better with experimental determinations than the one used above; but this calculation has not any very great physical value, since white light does not consist of a succession of pulses of exactly the same shape, so what we have arrived at is only a a sort of mean shape of the pulses.

## EXAMPLES.

1. If
$$\int_0^{\pi} F(x) \cos nx dx = \varphi(n)$$
prove that
$$\int_0^{\infty} \varphi(x) \cos nx dx = \tfrac{1}{2}\varphi(0) + \varphi(1) \cos n + \varphi(2) \cos 2n + \varphi(3) \cos 3n + \ldots$$
provided $0 < n < \pi$.

[For within this range each side is equal to $\frac{\pi}{2} F(n)$.]

2. Prove, similarly, that if
$$\int_0^{\pi} F(x) \sin nx dx = \psi(n)$$
then
$$\int_0^{\infty} \psi(x) \sin nx dx = \psi(1) \sin n + \psi(2) \sin 2n + \psi(3) \sin 3n + \ldots$$
provided that $0 < n < \pi$.

3. Verify the result in Example 1 when $F(x) = \pi - x$.

4. Prove that the energy, lying between the frequencies $\alpha/2\pi$ and $(\alpha + d\alpha)/2\pi$, in the pulse given by $f(t) = \dfrac{c}{c^2 + t^2}$ is proportional to $e^{-2c\alpha}$.

5. Prove, similarly, that for the pulse $f(t) = e^{-p^2t^2}$, the distribution of energy is proportional to $e^{-\alpha^2/2p^2}$.

6. Prove, similarly, that for the pulse $f(t) = \operatorname{sech} pt$, the distribution of energy is proportional to $\operatorname{sech}^2 \dfrac{\pi\alpha}{2p}$.

7. Show that the distribution of energy in the pulse, which is equal to unity from $t = -1$ to $t = 1$ and to zero outside these limits, is proportional to $\left(\dfrac{\sin \alpha}{\alpha}\right)^2$.

8. Show that the distribution of energy in the pulse, which is zero when $t$ is negative and equal to $e^{-kt} \sin pt$ when $t$ is positive (where $k$ is very small compared with $p$), is very approximately given by
$$\frac{1}{k^2 + (\alpha - p)^2}.$$

Hence note that a damped oscillation, when examined in a perfect spectroscope, will not show a perfectly sharp line, but will show a maximum of energy when $\alpha = p$, and the energy will be appreciable so long as $\alpha - p$ is not many times greater than $k$.

9. Show that the forms of the odd and even pulses, which will give a constant distribution of energy over the frequencies between $(\alpha - \delta)/2\pi$ and $(\alpha + \delta)/2\pi$, are

$$f(t) = \left(\frac{\sin \delta t}{t}\right) \sin \alpha t$$

and

$$f(t) = \left(\frac{\sin \delta t}{t}\right) \cos \alpha t$$

respectively.

10. The energy lying between the frequencies $\alpha/2\pi$ and $(\alpha + d\alpha)/2\pi$ is proportional to $(\alpha^{2p-2}e^{-2b\alpha})d\alpha$; prove that the forms of the odd and even pulses to give this distribution of energy are

$$f(t) = \frac{\sin (p \tan^{-1} t/b)}{(b^2 + t^2)^{p/2}}$$

and

$$f(t) = \frac{\cos (p \tan^{-1} t/b)}{(b^2 + t^2)^{p/2}}$$

respectively.

11. If $f(t)$ represents any convergent pulse, prove that the succession of pulses

$$\ldots + f(t - 2\pi) + f(t) + f(t + 2\pi) + f(t + 4\pi) + \ldots,$$

is represented by the Fourier series

$$\frac{a_0}{2} + a_1 \cos t + \ldots$$
$$+ b_1 \sin t + \ldots$$

where

$$a_n = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos nt \, dt$$

and

$$b_n = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin nt \, dt.$$

# CHAPTER VIII

## THE SEARCH FOR PERIODICITIES IN OBSERVED DATA NOT PERIODIC

The physicist, astronomer, or meteorologist—especially the latter two—frequently obtain curves which are suspected of containing one or more periodic components. If the curve is found to be definitely periodic, and the observations have extended over a sufficient number of complete periods to establish the fact beyond doubt, recourse can be had to the methods of harmonic analysis previously given when as accurate an analysis of the curve can be obtained as its own accuracy permits of. More generally, however, the curve contains several harmonics the periods of which are not in simple ratios to one another, so that the period within which the motion recurs is very large—if, in fact, it recurs at all. For instance, a curve composed of two harmonics of periods 12 and 17 days has a period of 204 days, but if the ratio of the periods, instead of being 1·416666 . . . , was in error by about 1 part in 760, so that the ratio of the periods was actually 1·414213 . . . or $\sqrt{2}$, the "period," instead of being 204 days, would be infinite; since there is no common multiple of two numbers bearing this ratio to one another less than infinity. Thus to calculate a period in cases like this, where the periods of some of the harmonics are only known to a limited degree of accuracy, is to tread on far too uncertain a ground.

We have thus to determine the presence of a harmonic constituent in the curve, or set of observations,

in the absence of any indications of a definite period. This can be done by the application of Fourier's integral theorem explained in the previous chapter. Let the given curve or observations be represented by $f(t)$ from $t = -\mathrm{T}$ to $t = \mathrm{T}$, and let us apply the theorem to the pulse defined to be equal to $f(t)$ from $t = -\mathrm{T}$ to $t = \mathrm{T}$ and to be zero everywhere outside this range. Then we see from page 151 that the energy in this pulse lying between the frequencies $\frac{\alpha}{2\pi}$ and $\frac{\alpha + d\alpha}{2\pi}$ is proportional to

$$\left\{\left[\int_{-\mathrm{T}}^{\mathrm{T}} f(\lambda) \cos \alpha\lambda \, d\lambda\right]^2 + \left[\int_{-\mathrm{T}}^{\mathrm{T}} f(\lambda) \sin \alpha\lambda \, d\lambda\right]^2\right\} d\alpha \quad . \quad . \quad (1)$$

or, say, $\mathrm{R}^2 d\alpha$. These integrals can be evaluated by Simpson's rule or other suitable method of quadratures; such evaluation must be done for a fairly large number of values of $\alpha$.

If $\mathrm{R}^2$ is plotted against $\alpha$ the result is called the "Periodogram" of the original curve or set of observations. We thus see that the periodogram is nothing other than the distribution of energy in the spectrum of the pulse defined as being equal to the observations over the range for which they extend and zero outside that range. Practically it is advantageous to split any given function into its odd and even components; R for either of these functions is then either

$$2\int_0^{\mathrm{T}} f(\lambda) \cos \alpha\lambda \, d\lambda \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

or

$$2\int_0^{\mathrm{T}} f(\lambda) \sin \alpha\lambda \, d\lambda \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

It is further advantageous to plot $|\mathrm{R}|$, instead of R, against $\alpha$ and to call *these diagrams* the periodograms of the functions in question.

We shall now show that the periodogram possesses pronounced maxima at each value of $\alpha$ which corresponds to a harmonic constituent in $f(t)$ provided that the range of observations extend over several such periods. As an example, let us suppose $f(t)$ consists of the harmonic $c \sin pt$ only and that the observations extend from $-\frac{N\pi}{p}$ to $+\frac{N\pi}{p}$ so that they cover N periods. In this case the first integral in (1) vanishes and we have

$$\begin{aligned} R &= 2c\int_0^{N\pi/p} \sin p\lambda \sin \alpha\lambda \, d\lambda \\ &= 2c\left\{\frac{\sin(p-\alpha)N\pi/p}{p-\alpha} - \frac{\sin(p+\alpha)N\pi/p}{p+\alpha}\right\} \quad . \quad . \quad (2) \end{aligned}$$

The first term attains its numerical maximum value of $2cN\pi/p$ when $\alpha = p$, while the second term necessarily lies within the limits $\pm 2c/(p+\alpha)$, and lies within the limits $\pm c/p$ in the neighbourhood of $\alpha = p$; that is, it is only $\pm 1/2N\pi$ of the maximum of the first term. The first term vanishes when $(p-\alpha)N\pi/p$ is a multiple of $\pi$ other than zero: that is, it vanishes when $\alpha = p(1 \pm 1/N)$, or $\alpha = p(1 \pm 2/N)$, etc. It will have subordinate numerical maxima at roughly half-way between these points; the value of the maxima when $(\alpha - p)/p = \pm(n + \frac{1}{2})/N$ being

$$\frac{2c}{(n+\frac{1}{2})p/N}$$

approximately, which is $\frac{1}{(n+\frac{1}{2})\pi}$ of the maximum when $\alpha = p$.

Fig. 22 shows a graph of $|R|$ when $N = 5$; the extent to which the values of $|R|$ are uncertain, owing to the neglect of the second term in (2), is shown by the distances between the two dotted curves

and the line midway between them. It is clear that such a relatively small uncertainty can do nothing to mask the subordinate maxima, let alone the very pronounced central maximum.

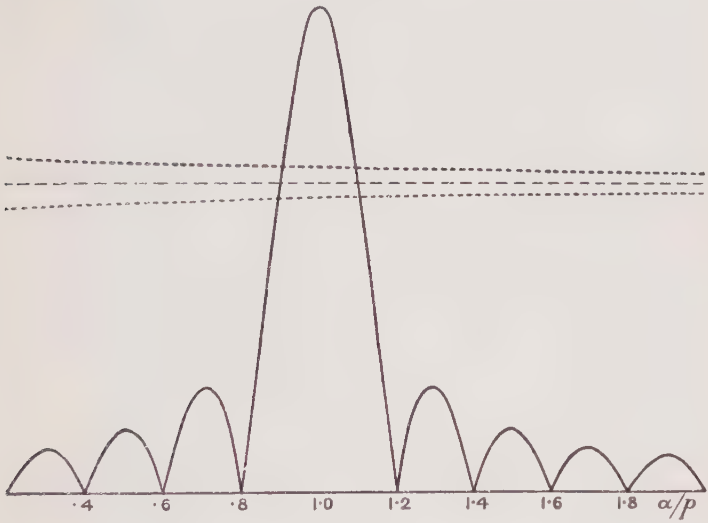


FIG. 22.

If we had considered the harmonic $c \cos pt$ instead of $c \sin pt$, we should have had

$$R = 2c\int_0^{N\pi/P} \cos p\lambda \cos \alpha\lambda \, d\lambda\,;$$

which would have given the same as (2) with merely the sign of the second term changed, so that the graph of $|R|$ would still lie within exactly the same limits as previously. It should also be noted that all these results hold whether N is an integer or not.

Let us now suppose that $f(t)$ contains two harmonics of different periods, say, $c \sin pt + c' \sin p't$. If the two periods $p$ and $p'$ are widely separated there

will be no difficulty in recognising the two central maxima corresponding to $\alpha = p$ and $\alpha = p'$. If $p$ and $p'$ are nearly equal, the two maxima become confused; but if the amplitudes of the two harmonics are at all comparable, separation of the periods is always *quite easy* when the two central peaks stand just clear of one another: that is when $(p' - p)/p = \pm 2/N$; or when the number of periods of $p'$ comprised within the range of observations differs from the number of periods of $p$ (which is N) by two. Hence to separate harmonics with nearly equal periods an extended series of observations is necessary.

Separation is generally easy when the maximum of one central peak coincides with the zero of the other; so that one of the harmonics has only one more period in the range of the observations than the other. In this case, if $c$ and $c'$ are of the same sign and comparable, the two central peaks will combine into a single peak whose width is three times that of the subordinate peaks. The subordinate peaks for the two periods will, in this case, reinforce one another and so will be prominent. If $c' = -c$ the peak for $p'$ would appear negative if we plotted R but since we plot $|R|$ it will readily be seen that the combination of the two peaks will give two equal peaks each of a width 1·5 times the width of the subordinate peaks. In this case the two systems of subordinate peaks will largely neutralise one another and so may be indistinguishable; nevertheless their width is known theoretically (viz. a variation of $1/N$ in the ratio $\alpha/p$), and the presence of two equal peaks each 1·5 times this theoretical width is conclusive proof of the presence of two harmonics of nearly equal periods and of amplitudes of opposite signs.

In our search for periodicities by this method we are not interested in the exact shape of the graphs: we merely use them for the positions of their maxima:

it follows that we could use instead any other curve which possessed equally distinct maxima in the same positions. A much more easily obtained curve, and one which is quite as satisfactory, can be found by replacing the $\cos \alpha\lambda$ and $\sin \alpha\lambda$ factors in the troublesome integrals

$$\int_0^T f(\lambda) \cos \alpha\lambda \, d\lambda \text{ and } \int_0^T f(\lambda) \sin \alpha\lambda \, d\lambda$$

by another periodic curve of the same period which will make the integrations easier. Such a curve is that

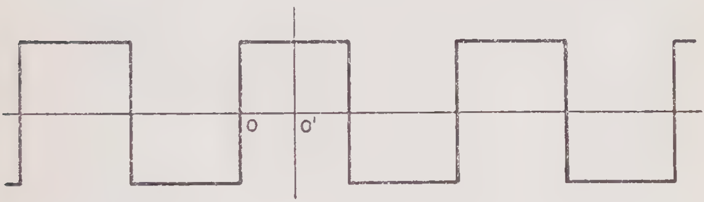


FIG. 23.

in Fig. 23, which is alternately $+1$ for half a period and then $-1$ for half a period. When the origin is chosen so that it is an odd function, its equation is

$$\phi_1(\alpha\lambda) = \frac{4}{\pi}\left\{\sin \alpha\lambda + \frac{\sin 3\alpha\lambda}{3} + \frac{\sin 5\alpha\lambda}{5} + \ldots\right\}$$

and when arranged as an even function, it is given by

$$\phi_2(\alpha\lambda) = \frac{4}{\pi}\left\{\cos \alpha\lambda - \frac{\cos 3\alpha\lambda}{3} + \frac{\cos 5\alpha\lambda}{5} - \ldots\right\}$$

Substituting $\phi_1(\alpha\lambda)$ for $\sin \alpha\lambda$ in the second of the given integrals gives us

$$\left[\int_0^{\pi/\alpha} - \int_{\pi/\alpha}^{2\pi/\alpha} + \int_{2\pi/\alpha}^{3\pi/\alpha} - \ldots \pm \int^T\right] f(\lambda) d\lambda \quad . \quad . \quad . \quad (3)$$

while the other one becomes

$$\left[\int_0^{\pi/2\alpha} - \int_{\pi/2\alpha}^{3\pi/2\alpha} + \int_{3\pi/2\alpha}^{5\pi/2\alpha} - \ . \ . \ . \ \pm \int^{T}\right] f(\lambda)d\lambda \ . \ . \ . \quad (4)$$

Each of these component integrals is very easily obtained, since we have the graph of $f(\lambda)$ (or can easily get it by plotting the observations) and from this we can read off the ordinates required in evaluating the integrals by Simpson's rule. How the graphs of $|R|$ are modified by using these functions $\phi_1$ and $\phi_2$ instead of the sine and cosine may easily be seen from their harmonic analysis. When using the function $\phi_1(\alpha\lambda)$ instead of $\sin \alpha\lambda$, the first term in (2) becomes

$$\frac{4}{\pi}\left\{\frac{\sin(p-\alpha)N\pi/p}{p-\alpha} + \frac{1}{3}\,\frac{\sin(p-3\alpha)N\pi/p}{p-3\alpha} + \ . \ . \ . \right\}$$

instead of

$$\frac{\sin(p-\alpha)N\pi/p}{p-\alpha},$$

the second term of (2) being just as negligible in this case as in the former case.

Thus the shape of the maximum in the neighbourhood of $\alpha = p$ is unaltered, but we have introduced into the graph for $|R|$ another maximum at $\alpha = p/3$ which is $\frac{1}{3}$rd of the height of the maximum at $\alpha = p$ and which need not correspond to any frequency in $f(t)$ at all. Similarly, there is another maximum at $\alpha = p/5$ which is $\frac{1}{5}$th of the maximum at $\alpha = p$ and which also does not exist in $f(t)$. It follows, therefore, that we can use this graph just as the correct one, provided we neglect these maxima at sub-multiple frequencies. Should, however, the maximum at $\alpha = p/3$ be much more than $\frac{1}{3}$rd of the maximum at $\alpha = p$, it is clear that a harmonic of the frequency corresponding to $\alpha = p/3$ must exist in $f(t)$, since

there is more of this frequency present than we have introduced by using the functions $\phi_1$ and $\phi_2$.*

We may vary the functions $\phi_1$ and $\phi_2$, so as to reduce still further the labour of evaluating the integrals involved, without affecting their effectiveness in detecting the various harmonic constituents. If we add unity to each of these functions, for instance, the result will only be to add to $|R|$ the constant quantity $\int_{-T}^{T} f(\lambda)d\lambda$; and will therefore make no difference to the peaks of $|R|$; yet it will have reduced the labour of finding $|R|$ to one half. Hence we may use for $\phi_1$ and $\phi_2$ functions which are zero for half a period and unity for the other half. It can readily be seen that we can go further in this direction and use functions which are zero for three quarters of a period and equal to unity for the other quarter. Going to the limit, we see that we can use functions which are zero throughout the whole period, save in the immediate neighbourhood of one point. In this case the trouble of integrating $\phi_1(\alpha\lambda) \cdot f(\lambda)$ and $\phi_2(\alpha\lambda) \cdot f(\lambda)$ is

* The student of optics will readily see the optical interpretation of this. The first periodogram corresponds to the analysis of a pulse by a grating which only produces a spectrum of one order—the first. The second one corresponds to a grating which not only produces a first order spectrum but a third order one of $\frac{1}{9}$th the energy of the first one and a fifth order of $\frac{1}{25}$th the energy and so on, the image in the third order underlying the image in the first order corresponding to $\frac{1}{3}$rd the frequency or three times the wave-length. No diffraction grating, only giving one order of spectrum, has ever been able to be constructed, although it is theoretically possible. We *have* to be content with one behaving more or less in the second manner. There is, similarly, no reason why we should insist on employing the analogue of the first case in constructing the periodogram. The student may also note that if the intensity of the image in the first case is unity, the intensities of the different order images in the second case are $8/\pi^2$, $8/9\pi^2$, $8/25\pi^2$ . . . which add up to unity.

done away with, as the result simply reduces to the sum of those values of $f(\lambda)$ for which the function $\phi_1$ or $\phi_2$ is not zero. This is made the basis of the practical method described in the next Section.

From the above we deduce the following method for searching for periodicities in any given series of observations. We take the origin of time at the middle of the range and represent the observations as the sum of an odd and an even function by the formula

$$f(t) = \frac{f(t) - f(-t)}{2} + \frac{f(t) + f(-t)}{2}.$$

For *each* of these functions we plot $|R|$ and find its central maximum peaks. Each of these corresponds to a definite harmonic in the series of observations. In this manner all cosine harmonics are found from the even function and all sine harmonics from the odd function.

Having found from the two graphs for $|R|$ the frequencies that exist in $f(t)$, we must next find their amplitudes and phases. Let $\alpha_1$ be the value of $\alpha$ corresponding to any frequency present in $f(t)$. Let $M_1$ be the central numerical maximum at $\alpha_1$ of the graph for $|R|$ constructed with the use of the $\cos \alpha\lambda$ factor for the even component of $f(t)$, and let $M_1'$ similarly denote the maximum for the odd component of $f(t)$ constructed in a similar manner with $\sin \alpha\lambda$.

Then if $a_1 \cos \alpha_1 t + b_1 \sin \alpha_1 t$ are the two corresponding harmonics in $f(t)$, we have from (2), (page 158),

$$M_1' = 2b_1 \frac{N\pi}{\alpha_1}.$$

But $2N\pi/\alpha_1$ is the total range of time, T, over which the observations extend. Hence we have the simple relation

$$b_1 = M_1'/T.$$

Similarly, $a_1 = M_1/T$.

If the graphs for $|R|$ had been constructed with the

aid of the functions $\phi_1(\alpha\lambda)$ and $\phi_2(\alpha\lambda)$ and if $M_1'$ and $M_1$ had been the values of their central maxima at $\alpha_1$ respectively we should have had

$$a_1 = \pi M_1/4T \quad . \quad . \quad . \quad . \quad (5)$$

and

$$b_1 = \pi M_1'/4T\,* \quad . \quad . \quad . \quad . \quad (6)$$

We must similarly write down the harmonic terms corresponding to any other plainly indicated central maxima flanked by their subordinate maxima and minima which occur in either of the $|R|$ graphs, neglecting, as we have explained above, central maxima corresponding to sub-multiple frequencies if the functions $\phi_1(\alpha\lambda)$ and $\phi_2(\alpha\lambda)$ have been used unless such maxima are unduly prominent.

Having found in this manner a number of periodicities in $f(t)$ and the amplitudes of the sine and cosine components of each of them, we may subtract the sum of all these terms from $f(t)$, when we shall be able to see whether the residue consists of periodic terms whose periods were outside the region tested, or whether it consists of a non-periodic function, or whether it appears to be nothing more than the errors of the given observations.

### *A Simple Way of Applying the Method in Practice.*

The foregoing method may often, in practice, be greatly simplified—so much so that it hardly seems to depend on the above theory—by making use of the fact that if we add the values of a function for equidistant intervals, all harmonics, save those for which the interval is a period, tend to cancel out.†

Let us suppose that the given observations can be

* Similar easily obtained results will hold when any other periodic functions are employed in place of $\varphi_1$ and $\varphi_2$.

† If the accuracy of the given observations is poor they should be plotted, and smoothed values read off from the graph in the ordinary way.

represented by several harmonic terms together with a slowly varying residual (if any), so that we may write:

$$f(t) = c_1 \cos(\alpha_1 t - \gamma_1) + c_2 \cos(\alpha_2 t - \gamma_2) + \ldots + \psi(t).$$

If we form the sum

$$f(t) + f\left(t + \frac{2\pi}{\alpha}\right) + f\left(t + \frac{4\pi}{\alpha}\right) + \quad . \quad . \qquad (7)$$

when $\alpha = \alpha_1$ taking as many terms, say N, in the sum as the extent of the observations permit, the first harmonic will become $Nc_1 \cos(\alpha_1 t - \gamma_1)$. The other harmonics will be in different phases in different terms of this sum unless their frequency is a multiple of $\alpha_1$; and so their resultant will in general be small compared with that of the harmonic $\alpha_1$. The effect of any harmonic, say, $\alpha_2$, will *not* be small if its extreme phases in the sum (7) do not differ by more than $\pi$ from one another, for then every term has a positive component along the line of the mean phase. This will be so whenever $\alpha_2\left(\frac{2\pi}{\alpha_1}\right)$ differs from a multiple of $2\pi$ by less than $\pi/N$; that is, when $\alpha_2/\alpha_1$ differs from an integer by less than $1/2N$. On the other hand, if the total range of the observations—that is, $2\pi N/\alpha_1$—is an integral multiple of the period of any other harmonic, such harmonic will absolutely cancel out in the sum unless its period is also a sub-multiple of $\alpha_1$. The effect of the slowly varying function $\psi(t)$ on the sum considered will only be to add to the harmonic $Nc_1 \cos(\alpha_1 t - \gamma_1)$ another slowly varying function which can do little to obscure it.

If this sum be plotted against $t$, the result is thus approximately the sine curve $Nc_1 \cos(\alpha_1 t - \gamma_1)$ plus something which is approximately constant. The amplitude of this harmonic can be found by measuring the total variation of height, V, of the graph which is equal to $2Nc_1$; also, if the position of the maximum value is at $t = t_1$, we have $\gamma_1 = \alpha_1 t_1$.

If this curve is not very approximately sine shaped, the presence of harmonics whose frequencies are multiples of $\alpha_1/2\pi$ are indicated. If this is so, these harmonics can be determined by subjecting this curve to the usual process of harmonic analysis.

The only difficulty about this method of analysis is to know when the value we take for $2\pi/\alpha$ in forming the sum (7) agrees with the period of one of the harmonics present in $f(t)$.

If we use a value of $\alpha$ slightly different from $\alpha_1$ the term $c_1 \cos(\alpha_1 t - \gamma_1)$ will not be in the same phase in all the terms of (5); and if $2\pi N\alpha_1/\alpha$ were equal to $(N \pm 1)2\pi$, the sum of all these terms would vanish, since their phases would be uniformly distributed over a range of $2\pi$. So it follows that the range of variation, V, has a central maximum when $\alpha = \alpha_1$. In fact, the variation is clearly according to the now familiar curve $y = \frac{\sin x}{x}$; or if we always consider V as positive (and it is not easy to tell when it should be considered as negative) according to the curve $y = \left|\frac{\sin x}{x}\right|$.

We accordingly work out the sum (7) for several different values of $t$ for each different value of $\alpha$ taken and plot the total range of variation V for changes of $t$ against $\alpha$ (or $2\pi/\alpha$) and pick out each central maximum. When drawing a graph through the points of this curve, unless we clearly see that one particular point lies right *at* the peak of one of the central maxima, we must re-work the sum (7), for the value of $\alpha$ which the graph indicates will be the highest point of the peak; for since the peaks are steep (the steepness increases with the time over which the observations extend), a point only a little way off the true centre may give a value of V much below the maximum value which is sought.

Having found these peaks, we can easily write down

the harmonic term corresponding to each peak. Subtracting the sum of these terms from $f(t)$, we can then examine the remainder and see whether it is constant, or some slowly varying (generally a decaying) function; or whether it consists of harmonic terms whose periods lay outside the region examined, or perhaps were missed, by taking too great a gap between the different values of $\alpha$ in the graph for V.

## An Illustrative Example of the Search for Periodicities.

We will suppose that some phenomena have been observed at 120 equal intervals of time (which we will call one day) with the following results:

| Day. | $f(t)$. | Day. | $f(t)$. | Day. | $f(t)$. | Day. | $f(t)$. |
|---|---|---|---|---|---|---|---|
| 0 | 19 | 31 | 3 | 62 | 24 | 93 | 14 |
| 1 | 25 | 32 | 9 | 63 | 25 | 94 | 24 |
| 2 | 28 | 33 | 17 | 64 | 25 | 95 | 34 |
| 3 | 30 | 34 | 27 | 65 | 25 | 96 | 32 |
| 4 | 30 | 35 | 37 | 66 | 17 | 97 | 29 |
| 5 | 29 | 36 | 36 | 67 | 10 | 98 | 22 |
| 6 | 21 | 37 | 32 | 68 | 6 | 99 | 14 |
| 7 | 14 | 38 | 25 | 69 | 5 | 100 | 4 |
| 8 | 11 | 39 | 17 | 70 | 5 | 101 | 4 |
| 9 | 9 | 40 | 7 | 71 | 15 | 102 | 5 |
| 10 | 9 | 41 | 7 | 72 | 24 | 103 | 9 |
| 11 | 19 | 42 | 8 | 73 | 30 | 104 | 14 |
| 12 | 28 | 43 | 12 | 74 | 34 | 105 | 23 |
| 13 | 34 | 44 | 18 | 75 | 35 | 106 | 24 |
| 14 | 37 | 45 | 26 | 76 | 25 | 107 | 23 |
| 15 | 39 | 46 | 26 | 77 | 15 | 108 | 22 |
| 16 | 29 | 47 | 26 | 78 | 6 | 109 | 18 |
| 17 | 18 | 48 | 24 | 79 | 0 | 110 | 12 |
| 18 | 10 | 49 | 21 | 80 | −4 | 111 | 13 |
| 19 | 3 | 50 | 15 | 81 | 4 | 112 | 13 |
| 20 | 0 | 51 | 16 | 82 | 14 | 113 | 13 |
| 21 | 8 | 52 | 16 | 83 | 24 | 114 | 14 |
| 22 | 18 | 53 | 16 | 84 | 33 | 115 | 18 |
| 23 | 28 | 54 | 17 | 85 | 39 | 116 | 14 |
| 24 | 37 | 55 | 21 | 86 | 33 | 117 | 13 |
| 25 | 43 | 56 | 17 | 87 | 24 | 118 | 13 |
| 26 | 36 | 57 | 16 | 88 | 14 | 119 | 13 |
| 27 | 28 | 58 | 16 | 89 | 4 | 120 | 12 |
| 28 | 18 | 59 | 16 | 90 | −5 | | |
| 29 | 7 | 60 | 14 | 91 | −1 | | |
| 30 | −1 | 61 | 20 | 92 | 5 | | |

Here we see successive maxima at intervals of 10 or 11 days, so there is certainly a component harmonic whose period is about that. We will test first of all for periods of 8, 9, 10, 11, 12, 13 days respectively.

The work for the 8-day period is shown in the table below. We have only made use of the observations extending over 10 periods, since the observations do not permit of us using a greater number in the case of the longer periods.

TABLE A.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 19 | 25 | 28 | 30 | 30 | 29 | 21 | 14 |
| 11 | 9 | 9 | 19 | 18 | 34 | 37 | 39 |
| 29 | 18 | 10 | 3 | 0 | 8 | 18 | 28 |
| 37 | 43 | 36 | 28 | 18 | 7 | −1 | 3 |
| 9 | 17 | 27 | 37 | 36 | 32 | 25 | 17 |
| 7 | 7 | 8 | 12 | 18 | 26 | 26 | 26 |
| 24 | 21 | 15 | 16 | 16 | 16 | 17 | 21 |
| 17 | 16 | 16 | 16 | 14 | 20 | 24 | 25 |
| 25 | 25 | 17 | 10 | 6 | 5 | 5 | 15 |
| 24 | 30 | 34 | 35 | 25 | 15 | 6 | 0 |
| 202 | 211 | 200 | 206 | 191 | 185 | 178 | 188 |

The Range, V, is $211 - 178 = 33$.

Proceeding in this way for the other periods, we get the following results:

| Period | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|
| " V " | 33 | 32 | 269 | 35 | 202 | 38 * |

No satisfactory graph can be drawn from these irregular figures, so we must test similarly for intermediate periods, say, of 8·5, 9·5, 10·5, 11·5 and 12·5 days. To test for $8\frac{1}{2}$ days we write down in the first column the value of $f(t)$ for 0, 8, 17, 25, 34, 42, 51, 59, 68, 76 days respectively, and in the successive columns for 1 day greater than each of these numbers successively. The intervals between these numbers

* In this last case we can only test over a range of 9 periods and we get $V = 34$. We therefore increase the result in the ratio 10 : 9.

are 8 and 9 days alternately. Had a complete *graph* of $f(t)$ been available for the whole of the 120 days, we could have picked out ordinates at intervals of 8·5 days for this test, but ordinates at intervals of 8 and 9 days alternately are quite as satisfactory, since the difference of phase from the mean phase is only $2\pi/17$, the cosine of which is ·93.

Similarly, if we were testing for a period of $8\frac{1}{3}$ days,

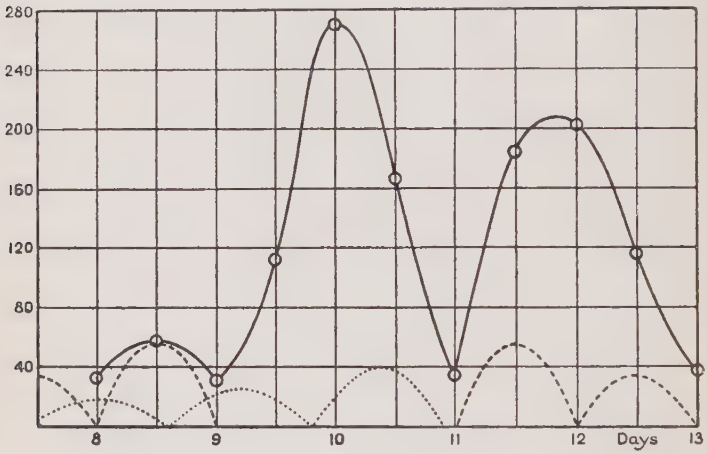


Fig. 24.

we should write down in the first column the values of $f(t)$ for $t = 0, 8, 16, 25, 33, 41, 50, \ldots$ successively, where the intervals are 8, 8, 9, 8, 8, 9 . . .

In this manner, we obtain the following results :

| | | | | | |
|---|---|---|---|---|---|
| Period | 8·5 | 9·5 | 10·5 | 11·5 | 12·5 |
| " V " | 58 | 112 | 167 | 184 | 116 |

Fig. 24 shows the graph obtained by plotting V against the period in days.

There is clearly one period of 10 days very approximately and another about 12 days. Since the position of this latter maximum is a bit uncertain, we calculate

V for a period of $11\frac{3}{4}$ days, which gives V = 199. We then infer a period of about 11·9 days for this second peak.

The heavy dotted curves show the secondary maxima corresponding to the peak for the 10-day period. This makes it clear that the distortion of the peak for the 12-day period is largely due to the first secondary maximum of the 10-day period peak. We may therefore take 12 days for the second period. We cannot possibly expect to find the period to a greater accuracy than this; for the difference is only 1 day in the length of 10 periods, which is only a difference of one-twelfth of a period in the whole range of the observations.

We next plot the sums of Table A for the 10- and 12-day periods against *t*.

For the 12-day period the sums are:

| *t*. | Sum. | *t*. | Sum. | *t*. | Sum. |
|---|---|---|---|---|---|
| 0 | 269 | 4 | 181 | 8 | 91 |
| 1 | 291 | 5 | 138 | 9 | 139 |
| 2 | 266 | 6 | 92 | 10 | 176 |
| 3 | 241 | 7 | 89 | 11 | 235 |

These when plotted give sensibly a sine curve of which the equation is

$$y = 100 \sin\left(\frac{2\pi t}{12} + \frac{\pi}{3}\right)$$

superposed on a gradually decaying function; but since each "sum" is the sum of ten items, the equation of the harmonic in $f(t)$ is

$$y = 10 \sin\left(\frac{2\pi t}{12} + \frac{\pi}{3}\right).$$

Since the range of observations used in obtaining this curve—120 days—is exactly 12 times the 10-day period, this curve is not at all affected by this latter period. It is different, however, with the test for the 10-day period. A range of 100 days only was used for this test, which is not a multiple of the 12-day

period. Since we have sufficient observations at our disposal, we re-calculate this test, using 12 ten-day periods instead of only 10. We then get the following figures :

| *t.* | Sum. | *t.* | Sum. | *t.* | Sum. |
|---|---|---|---|---|---|
| 0 | 75 | 4 | 310 | 8 | 187 |
| 1 | 133 | 5 | 369 | 9 | 127 |
| 2 | 191 | 6 | 310 | | |
| 3 | 252 | 7 | 248 | | |

On plotting these figures, we get the result shown in Fig. 25, which we recognise as a Class IV periodic function * whose equation is of the form

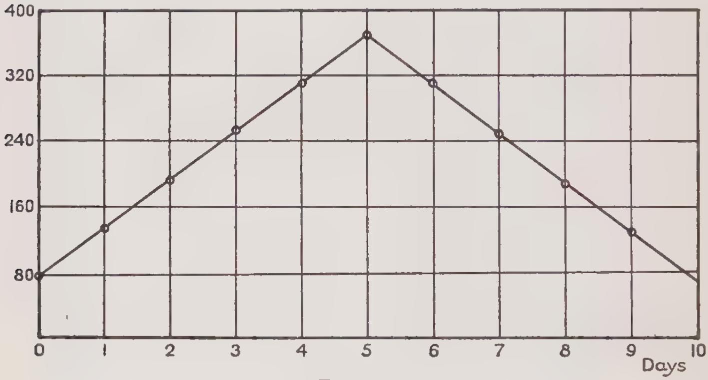


FIG. 25.

$$y = \mathrm{C} - \frac{8\mathrm{C}}{\pi^2}\left\{\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \ldots\right\}.$$

On determining the amplitude, which is as near 150 as possible, and dividing by 12, we find that the expression

$$12{\cdot}5 - \frac{100}{\pi^2}\left\{\cos\frac{2\pi t}{10} + \frac{1}{9}\cos\frac{6\pi t}{10} + \frac{1}{25}\cos\frac{10\pi t}{10} + \ldots\right\}$$

is a constituent of $f(t)$.

* Neglecting the gradually decaying function mixed up with it which is indicated by the ordinate for the 10th day being less than the initial ordinate.

We next subtract the sum of this term and the 12-day period term from $f(t)$ to study the residue. The values of both terms are very easily read off from the graphs we have previously constructed of them. We do this for every fifth day, thinking that sufficient. The following results are obtained:

| $t$. | Residue. | $t$. | Residue. |
|---|---|---|---|
| 0 | 10·3 | 65 | 5·0 |
| 5 | 9·0 | 70 | 5·0 |
| 10 | 9·0 | 75 | 5·0 |
| 15 | 9·0 | 80 | 4·7 |
| 20 | 8·7 | 85 | 4·0 |
| 25 | 8·0 | 90 | 3·7 |
| 30 | 7·7 | 95 | 4·0 |
| 35 | 7·0 | 100 | 4·0 |
| 40 | 7·0 | 105 | 3·0 |
| 45 | 6·0 | 110 | 3·3 |
| 50 | 6·3 | 115 | 3·0 |
| 55 | 6·0 | 120 | 3·3 |
| 60 | 5·3 | | |

Since $f(t)$ was only given in the first place to integers, these residues must be considered as uniformly decreasing. Plotting them reveals a curve of an exponential form which seems to decay to zero ultimately. The time of falling to $1/e$ of its initial value, which is 10, is seen, from the graph, to be about 100 days.

Hence the expression

$$10e^{-\cdot 01t} + 10\sin\left(\frac{2\pi t}{12} + \frac{\pi}{3}\right) + 12\cdot 5 - \frac{100}{\pi^2}\left\{\cos\frac{2\pi t}{10} + \frac{1}{9}\cos\frac{6\pi t}{10} + \ldots\right\}$$

represents the given observations to well within the accuracy to which they are given: viz., half a unit.

# CHAPTER IX

## A BRIEF HISTORICAL SURVEY

ALTHOUGH the purpose of this book is to impart such a knowledge of the subject that the reader shall be able to make practical use of his knowledge, a few historical facts about the development of the subject will probably be welcomed for their interest. The history of our subject is very simple: * so much is due to Fourier and so little to other people.

The fact that an analytic function could be represented in what we now call a Fourier series was known in 1777 to Euler (1707–1783), who obtained the coefficients in the same manner as we obtained them on page 22. With an analytic function, of course, there is no question about the *possibility* of the expansion.

Jean Baptiste Joseph Fourier (1768–1830) was, however, the first to assert that *any* finite *artificial* function could be represented, between any two finite limits, by such a series, and that the coefficients were given by the same formulæ as for analytic functions. Fourier's assertions were at first emphatically denied by the leading French mathematicians of the day, including Lagrange, and his proofs were regarded as inconclusive.

Fourier arrived at his results during the consideration of the solution of problems on the flow of heat;

* We are not now considering the complicated theoretical developments of the *theory* of Fourier's series which have been made in comparatively recent years and with which this *Practical Treatise* is entirely unconcerned.

and his results, though published previously, are fully incorporated in his famous *Théorie Analytique de La Chaleur* (1822), one of the great classics of Mathematical Physics, which even now no mathematician ever reads without feeling a genuine admiration for Fourier's genius, or ever refers to except in terms of praise. Fourier's work and Fourier's theorem will always remain fundamental in the solution of that great variety of important physical problems which can be formulated in terms of partial differential equations. No doubt Fourier's proofs of his theorem leave a modern mathematician rather doubtfully convinced; but Fourier's work is a striking illustration of the fact that the intuitions of a genius are of infinitely more value to science than the proofs of a pedant.

Fourier also gave the special application of his theorem to odd and to even periodic functions, and, further, the result now known as Fourier's Integral Theorem, which is what the ordinary theorem becomes when the period is made infinite. In fact, in nearly every way, Fourier showed a much clearer comprehension of the subject than many subsequent writers. Fourier alone, at first, grasped the fact that he had found a method of great importance. Applied to the analysis of analytic periodic functions only, the theorem was of no importance whatever: it was only an awkward way of effecting a transformation that could readily be effected by a little algebra, as in Chapter IV.

Poisson (1781–1840) and Dirichlet (1805–1859) were among the first to give a proof of Fourier's theorem by the direct summation of the terms of Fourier's series, Dirichlet's work being the more satisfactory, and rendering any further doubt about the validity of Fourier's expansion, for functions which satisfied his simple conditions, impossible.

Dirichlet proved that for any given value of $t$ the

sum of Fourier's series *was* $f(t)$, provided $t$ was not a point of discontinuity, and equal to $\frac{1}{2}\{f(t-0)+f(t+0)\}$ at such a point. Dirichlet's proof is reproduced in a simplified and abbreviated form on pp.136–8.

The fact that, at a point of discontinuity of a function, the sum of the series was really indeterminate within certain definite limits, and that the range of such indeterminateness was 1·179 times the abrupt change in magnitude of the function, seems only to have been noticed comparatively recently; though for ourselves we can hardly make out how it could have escaped detection in Fourier's day that the series

$$\frac{\sin t}{1}+\frac{\sin 2t}{2}+\ldots,$$

when $t=0$, can have any value that $\int_0^X \frac{\sin x}{x}\,dx$ can take; since everybody knew even then that the limit when $t=0$ of

$$t\{f(t)+f(2t)+f(3t)+\ldots+f(X)\}$$

was $\int_0^X f(x)dx$, and one has only to multiply both numerator and denominator of each term of the series by $t$ to make the result instantly obvious.

The subject of the harmonic analysis of a function defined by a set of points was not developed in any practical detail by these early mathematicians because the scientific observations requiring such treatment were not then extant. Lagrange (1736–1813), subsequent to his doubts over Fourier's results, obtained the equation of a finite sine series passing through the tops of a number of equidistant ordinates; while Poisson extended this result to a finite series including both sines and cosines and so gave the fundamental formulæ for all harmonic analysis with equi-distant

ordinates. Lagrange's and Poisson's results are exactly the same as those given in our Chapter V. No doubt Poisson, had he required to undertake practical harmonic analysis, would have invented Schedules for calculating the coefficients almost indistinguishable from those we have given. Several formulæ for interpolation for periodic functions, when the ordinates were not equi-distant, were given by Gauss (1777–1855). As far as we know, the complete list of these formulæ, with indications of the types of function to which they are respectively applicable, is now given for the first time at the end of Chapter V.

One would think it impossible for mathematicians to pay much attention to the harmonic analysis of functions defined by a set of equi-distant ordinates without soon arriving at the expressions giving the coefficients of the different harmonics in terms of the actual Fourier constants of the function. It is some surprise to us, therefore, that we have not seen these relations in print earlier than A. Clayton's article in the *Journal of the Institution of Electrical Engineers*, Vol. LIX, 1920–21, p. 491; though it seems inevitable that several workers must have obtained these relations for themselves prior to this.

Coming to the subject matter of Chapter VII, which is essentially due to Fourier, the late Lord Rayleigh (1842–1919) was the first to prove that if

$$\int_{-\infty}^{\infty} f(t) \cos at\, dt \equiv \mathrm{A}$$

and

$$\int_{-\infty}^{\infty} f(t) \sin at\, dt \equiv \mathrm{B},$$

then

$$\int_{-\infty}^{\infty} f(t)^2 dt = \frac{1}{\pi}\int_{0}^{\infty} (\mathrm{A}^2 + \mathrm{B}^2)\, da,$$

which is the extension to pulses, of the well-known

result that the energy of any periodic wave is the sum of the energy of its different harmonics.

Finally, we must state that the very simple method given at the end of Chapter VIII for the practical determination of harmonic constituents in a non-periodic function is the same as that given in Whittaker and Robinson's *Calculus of Observations* (1924). Although much attention has been given to this important practical problem, from the time of Lagrange to the present day, the theory of the problem that we have given shows that nothing simpler can be obtained than this; while at the same time it shows that more elaborate methods are not worth the extra labour of applying, and are no more necessary than is the ideal diffraction grating, which only gives a single order spectrum, to the experimenter in spectroscopy.

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